Probability Distributions for the Phase Jitter in Self-Timed Reconstructive Repeaters for PCM

By M. R. AARON and J. R. GRAY

(Manuscript received August 25, 1961)

Probability distributions for the timing jitter in the output of an idealized self-timed repeater for reconstructing a PCM signal are approximated. Primary emphasis is focused on self-timed repeaters employing complete retiming. In this case the probability distribution for the timing jitter reduces to the computation of the phase error in the zero crossings at the output of the tuned circuit excited by a jitter-free binary pulse train. It is assumed that the tuned circuit is mistuned from the pulse repetition frequency, and the individual pulses are either impulses or raised cosine pulses. Both random pulse trains and random plus periodic trains are considered. In general, the probability distributions are skewed in the direction of increasing phase error. The approach to the normal law in the neighborhood of the mean when the circuit Q becomes arbitrarily large is demonstrated. Results obtained from the analytical approach are compared with two computer methods for the case of random impulse excitation of a tuned circuit characterized by a Q of 125 and mistuning of 0.1 per cent. Excellent agreement between the three techniques is displayed. For no mistuning and raised cosine excitation two methods for computing the phase error are given and numerical results obtained from both techniques agree closely.

Some attention is given to an idealized version of a reconstructive repeater employing partial retiming and it is shown that the timing performance of such a repeater for random signals is very much inferior to the completely retimed repeater.

I. INTRODUCTION

Over the past several years the problem of maintaining pulse spacing within very close bounds in PCM transmission has received considerable attention both theoretically and experimentally. The effects of timing jitter in degrading repeater performance, in introducing distortion in

the decoded analog signal, and in enhancing the difficulty of dropping or adding several pulse trains in time have been documented.[1-8] Sources of mistiming in a self-timed reconstructive repeater are well catalogued and include: noise, crosstalk, mistuning, finite pulse width effects, and amplitude to phase conversion in nonlinear devices. The first four of these sources have been considered in various analyses of timing jitter in self-timed and separately-timed PCM repeaters. Amplitude to phase conversion in nonlinear circuits has received attention primarily from the experimental viewpoint.

The majority of the theoretical work to date has been concerned with timing errors in self-timed repeaters when the timing-wave extractor is a simple tuned circuit. For a random pulse train exciting the tuned circuit in the presence of noise and mistuning, results have been obtained for the mean displacement and the standard deviation of the zero crossings from their ideal location. This analysis is appropriate to repeaters employing complete retiming. These time displacements can also be considered as phase errors and we will use this terminology in what follows. If the probability density function for the phase error is normal, the mean and standard deviation are sufficient for a complete statistical description. In this paper we will show that in general the probability density function is not normal, and is inherently unsymmetrical about the mean.

An approximation to the probability density and the cumulative distribution for the phase error at the output of a mistuned resonant circuit will be derived for both random and random plus periodic pulse trains. A completely random pulse train is defined to be one in which pulses and spaces are equally likely. The individual pulses of the binary pulse train are assumed to be jitter free and are either impulses or raised cosine pulses. The approach to the normal law when the circuit Q is large is demonstrated. For a value of Q of 125, and a mistuning of 0.1 per cent from the pulse repetition frequency a comparison of numerical results obtained from the analytical approach and two computer methods is made. Agreement among the three approaches is excellent.

Our plan of attack is to place all of the manipulations required to specify the tuned circuit response to the most general pulse trains in the Appendix and concentrate on most of the probabilistic notions in the main body of the paper. Appendix A covers the response of the tuned circuit to a random or random plus periodic binary pulse train of arbitrary pulse shape, and Appendix B is concerned with the specialization to raised cosine pulses. Section II of the text deals with the terminology required, covers the tuned circuit response to impulses, and briefly

summarizes the results of Appendices A and B. In Section III, the probability density function for the phase error is derived. Section IV is devoted to the cumulative distribution function and Section V alludes to the semi-invariants that are required in the evaluation of the density and cumulative distribution functions. These semi-invariants are derived in Appendix C. The approach of the probability density function for the phase error to the normal law as the circuit Q becomes arbitrarily large is displayed in Section VI with the algebraic support relegated to Appendix D. The comparison of numerical results mentioned previously with other computer approaches is made in Section VII. For zero mistuning, but finite pulse width excitation, it can be shown that the probability distributions for the phase error can be related directly to the probability distribution for the timing wave amplitude. This is demonstrated in Section VIII. A discussion of further numerical results is given in Section IX. We consider an idealized model of a partially retimed repeater in Section X for purposes of comparison with the results of Section IX. A wrap-up of the procedures, results, and future work concludes the paper.

II. RESPONSE OF THE TIMING CIRCUIT

Before we go on to the general equation for the phase error due to finite pulse width and mistuning, we will specialize to impulse excitation of a simple tuned circuit characterized by its Q and mistuning from the pulse repetition frequency. This should provide the casual reader with some feel for how the more general equation for the phase error arises without going through the detailed manipulations of Appendices A and B. The procedure adopted in the analysis to follow is equivalent to that of H. E. Rowe.[2]

Assuming the input to the timing circuit to be a train of jitter-free unit impulses occurring at random with spacing T, the excitation may be represented as

$$f(t) = \sum_{n=-\infty}^{n=\infty} a_n \delta(t - nT), \tag{1}$$

where a_n is a random variable taking the values 0 or 1 with probability $\frac{1}{2}$,* $\delta(t - nT)$ is a unit impulse whose time of arrival is nT, and the spacing T is the reciprocal of the pulse repetition frequency f_r. For a parallel resonant circuit the impulse response is given by

* Unless otherwise specified, the case of equal likelihood will be considered in all calculations.

$$h(t) = A\, e^{-(\pi/Q)f_o t} \cos (2\pi f_o t + \varphi), \tag{2}$$

where

$$f_o = \frac{1}{2\pi} \sqrt{\frac{1}{LC} - \left(\frac{1}{2RC}\right)^2}, \qquad A = \frac{1}{2QC} \sqrt{4Q^2 + 1},$$

$$Q = 2\pi f_o RC, \qquad \text{and} \qquad \varphi = \tan^{-1} \frac{1}{2Q}.$$

Here f_o is the natural resonant frequency as distinguished from the steady-state resonant frequency $f_s = (1/2\pi)\sqrt{1/LC}$. Combining (1) and (2), the total response to all impulses occurring in time slots up to and including the one at $t = 0$ may be written as

$$F(t) = A \sum_{n=-\infty}^{n=0} a_n\, e^{-(\pi/Q)f_o(t-nT)} \cos [2\pi f_o(t - nT) + \varphi]. \tag{3}$$

This expression gives the output of the timing circuit for values of t in the interval between $t = 0$ and the arrival time of the next impulse. Rewriting (3) in the form of a carrier with both amplitude and phase modulation we get

$$F(t) = A\sqrt{x^2 + y^2}\, e^{-(\pi/Q)f_o t} \cos [2\pi f_o t + \varphi + \theta], \tag{4}$$

where

$$\theta = \tan^{-1} \frac{y}{x},$$

$$x = \sum_{n=0}^{\infty} a_n\, e^{-(\pi/Q)f_o nT} \cos 2\pi f_o nT, \quad \text{and}$$

$$y = \sum_{n=0}^{\infty} a_n\, e^{-(\pi/Q)f_o nT} \sin 2\pi f_o nT.$$

In the above x and y represent the in-phase and quadrature components of the response. If the tank could be tuned exactly to the pulse repetition frequency ($f_o \equiv f_r \equiv 1/T$), then the phase modulation would disappear and the amplitude modulation would be dependent on x alone. In practical applications this is not possible and the phase shift θ does occur. If we denote the fractional mistuning $\Delta f/f_r$ by k, we may write f_o in terms of f_r as follows

$$f_o = f_r(1 + k).$$

In this case (4) becomes, neglecting k with respect to unity in the exponential term

$$F(t) = A\sqrt{x^2 + y^2}\, e^{-(\pi/Q)f_r t} \cos\,[2\pi f_r(1 + k)t + \varphi + \theta], \tag{5}$$

with

$$x = \sum_{n=0}^{\infty} a_n\, e^{-(\pi/Q)n} \cos 2\pi kn,$$

$$y = \sum_{n=0}^{\infty} a_n\, e^{-(\pi/Q)n} \sin 2\pi kn,$$

and

$$\theta = \tan^{-1} y/x.$$

To illustrate the relationship between the timing deviation t_d and the phase error θ, it is assumed that repeater delays have been adjusted so that the timing wave supplied to the regenerator in the absence of mistuning is properly aligned with the signal impulses in the information-bearing channel. In this case, the negative-going zero crossing occurring ideally at $t_o = T/4$ determines the instant of regeneration. When mistuning is present this zero crossing is displaced such that it occurs at the instant $t_o' = T(\frac{1}{4} - \theta/2\pi)$. The difference $t_o - t_o'$ will then give the timing deviation which, expressed as a fractional part of the pulse spacing, is

$$\frac{t_d}{T} = \frac{\theta}{2\pi}. \tag{6}$$

From (6) and the definition of θ, the phase error corresponding to the timing deviation is related to the random variables x and y by

$$\theta = \tan^{-1} \frac{y}{x}. \tag{7}$$

In deriving (7) it should be recalled that only the incidental approximation $k \ll 1$ has been made. When we consider a binary pulse train in which the pulses representing the binary "one" are of arbitrary pulse shape, it is necessary to make other approximations to arrive at a tractable expression for the phase error. Furthermore, the excitation encompasses the infinite past as well as the tails of succeeding pulses to accommodate driving pulses that may overlap or are not time limited. The most general result given by (59) is an extension along two lines of Rowe's relationship for the timing jitter in the output of the tuned circuit due to mistuning and finite pulse width. First, the results are applicable to arbitrary pulse shape. Secondly, our relationship for the

phase error is based on a different approximation in the case of finite width pulses.

In appendix B we specialize to the case of raised cosine pulses in order to make use of some of Rowe's results. For this case the phase error is given by (73) and takes the form

$$\theta = \frac{y + a}{x + b} + c, \tag{8}$$

where a, b, and c are constants that depend upon Q, k, and the pulse width T/s of the raised cosine pulse. x and y are correlated random variables that depend upon Q, k, and the pulse pattern. They are defined below (5) with the additional constraint that $a_o = 1$ when we consider finite width pulse; i.e., a pulse definitely occurs at the origin. In our notation, a positive phase error corresponds to the zero crossing of interest occurring prior to the reference. The largest pulse width we consider is $1.5T$. This avoids the necessity of considering the effect of the presence or absence of a following pulse on the negative-going zero crossing of interest. Similarly, for positive-going zero crossings we do not have to use special methods for considering the occurrence or non-occurrence of a preceding pulse. This is not a serious analytical restriction, since larger pulse widths can be handled by the machinery provided in Section A-4. As a practical matter in the design of a self-timed reconstructive repeater for operation in a long repeater chain, wider pulses would introduce intolerable phase jitter. In the following, we will also neglect the constant c in (8), since it is independent of pulse pattern and can in principle be compensated for in either the timing path or information-bearing path in a self-timed reconstructive repeater.

III. PROBABILITY DENSITY FOR THE PHASE ERROR

3.1 *Preliminaries*

From the above, the random variable of interest is

$$\theta = \frac{y + a}{x + b} = \frac{y_1}{x_1}. \tag{9}$$

To determine the probability density $p(\theta)$ or the cumulative distribution $F(\theta)$, we consider the joint probability density of the correlated random variables x_1 and y_1, $p(x_1, y_1)$. $F(\theta) = \Pr(y_1/x_1) \leqq \theta)$, which may be written

$$F(\theta) = \int_0^\infty dx_1 \int_{-\infty}^{\theta x_1} dy_1 p(x_1, y_1) + \int_{-\infty}^0 dx_1 \int_{\theta x_1}^\infty dy_1 p(x_1, y_1).$$

Differentiation of $F(\theta)$ with respect to θ plus rearrangement yields

$$p(\theta) = \int_0^\infty x_1 p(x_1, \theta x_1)\, dx_1 + \int_0^\infty x_1 p(-x_1, -\theta x_1)\, dx_1 . \qquad (10)$$

Therefore if $p(x_1, y_1)$ is known, $p(\theta)$ can be determined by integration. As is typical of this class of problems when x_1 and y_1 are not correlated normal variables, the exact determination of $p(x_1, y_1)$ is rarely obtainable. Therefore, we find it essential to proceed along approximate lines.

We can write the characteristic function $\varphi(u,v)$ for $p(x_1, y_1)$ as

$$\varphi(u,v) = \int_{-\infty}^{\infty} dx_1 \int_{\infty}^{\infty} dy_1 e^{i(ux_1+vy_1)} p(x_1, y_1) . \qquad (11)$$

If we take the partial derivative of (11) with respect to u, evaluate it at $u = -\theta v$, divide both sides by $2\pi i$, and integrate over v from $-\infty$ to ∞, we get

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\partial \varphi(u,v)}{\partial u} \bigg|_{u=-\theta v} dv = \frac{1}{2\pi} \int_{-\infty}^{\infty} dv \int_{\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dy_1 x_1 e^{iv(y_1 - \theta x_1)} p(x_1, y_1) .$$

When we interchange the order of integration to integrate over v first,

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\partial \varphi(u,v)}{\partial u} \bigg|_{u=-\theta v} dv = \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dy_1 x_1 \delta(y_1 - \theta x_1) p(x_1 y_1) ,$$

where $\delta(y_1 - \theta x_1)$ is the Dirac delta function. Integration over y_1 then results in

$$\begin{aligned} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\partial \varphi(u,v)}{\partial u} \bigg|_{u=-\theta v} dv &= \int_{-\infty}^{\infty} x_1 p(x_1, \theta x_1)\, dx_1 \\ &= \int_0^\infty x_1 p(x_1, \theta x_1)\, dx_1 - \int_0^\infty x_1 p(-x_1, -\theta x_1)\, dx_1 . \end{aligned} \qquad (12)$$

A comparison of (10) with (12) reveals that they are equal provided that x_1 is always positive, in which case $p(-x_1, -\theta x_1)$ is zero. Under this condition[9]

$$p(\theta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\partial \varphi(u,v)}{\partial u} \bigg|_{u=-\theta v} dv. \qquad (13)^*$$

In the following we will use (13) to approximate $p(\theta)$; before doing so we make a few remarks about the range of the random variables x_1 and θ.

* The result in (13) is given as an exercise for the reader on p. 317 of Ref. 9.

3.2 *Minimum Values of x_1 and y_1*

Our comments in this section will largely be confined to the case of impulse excitation in which case $x_1 = x$ and $y_1 = y$, where x and y are defined following (5). From the definition of x it can be seen that it attains its minimum value for the set of $a_n = 1$ in which the argument of $\cos 2\pi kn$ is in the second and third quadrants (modulo 2π). With this pulse pattern it is easily shown that

$$x_{\min} = -\frac{\beta \sin 2\pi k e^{-(\pi/4kQ)}(1 + e^{-(\pi/2kQ)})}{(1 - e^{-(\pi/2kQ)})(1 - 2\beta \cos 2\pi k + \beta^2)} = -2\bar{y}\,\frac{e^{-(\pi/4kQ)}}{(1 - e^{-(\pi/2kQ)})}$$

where $\beta = e^{-(\pi/Q)}$ and $\bar{y}$ = average value of y (from Appendix D). For the values of k and Q that we consider, namely kQ less than about 0.1 and $Q \geqq 100$, an excellent approximation for $x_{\min}$ is

$$x_{\min} = -2\bar{y}e^{-(\pi/4kQ)}.$$

When kQ is fixed at 0.1,

$$x_{\min} \doteq \frac{4kQ^2}{\pi}\, e^{-2.5\pi}$$

and for $Q = 100$, $x_{\min} = -0.005$. The ratio $x_{\min}/\bar{x}$, where $\bar{x}$ = average value of x, can be shown to be

$$\frac{x_{\min}}{\bar{x}} \doteq -4kQ\, e^{-(\pi/4kQ)},$$

which for $kQ = 0.1$ is -0.00016, or very close to zero. Based on unpublished work of one of the authors, the probability of $x/\bar{x}$ of even going negative is so remote as to be completely unimportant and decreases with increasing Q for kQ fixed.

Another interesting way of looking at the probability of x becoming negative is to consider the probability of pulses occurring in the first quadrant of the argument of $\cos 2\pi kn$ to constrain the minimum value of x to zero. This can occur in any of several ways. One possibility is to choose a single pulse (a single $a_n = 1$) in the sector of the first quadrant bounded by $n = 0$ and the largest integral value of n that satisfies

$$\beta^n \cos 2\pi kn > |\, x_{\min} \,|.$$

For $Q = 100$ and $kQ = 0.1$, the above is satisfied for a value of n that is less than about 148. The probability of at least one pulse in this range of n is $1 - (1 - p)^{148}$ which is about $1 - 10^{-18}$ for equally likely pulses and spaces. Therefore, x is positive with probability very close to unity.

For increasing values of Q, with kQ fixed at 0.1, the probability that x is >0 approaches unity even more closely.

By an argument that parallels the above, the probability that $y < 0$ for $k > 0$ and impulse excitation is very small. Similarly, probability $y > 0$ for $k < 0$ is extremely small.

For raised cosine excitation, $x_{\min}$ is increased by $1 + b$, which for the pulse widths considered herein is always >0.25, thereby making $x_{\min}$ positive for the Q's of interest to us. We also note that long strings of zeros as required in attaining $x_{\min}$ cannot be tolerated in a PCM repeater with a simple tuned circuit timing extractor, since the timing wave amplitude would fall well below the point at which it would be useful in the repeater. A higher minimum on the timing wave amplitude can be assured by constraining the transmitted pulse train to avoid such long strings of spaces.[7] In this paper we simulate this constraint by the introduction of a forced periodic pattern of pulses in the otherwise random train. This serves to increase $x_{\min}$ and decrease the range of θ as we shall see below and in Sections VII and VIII.

3.3 *Range of* θ

For random impulse excitation, it is apparent from (5) that θ is unbounded when we choose a single $a_n = 1$ for n large and all the rest zero. However, with $a_o = 1$ and the values of Q we consider, x is always positive, and from the results of Section 3.2 θ is essentially confined to $(0, \pi/2)$ for $k > 0$ and $[0, -(\pi/2)]$ for $k < 0$. In the following we seek tighter bounds under the practically important case $a_o = 1$. Experimentally, $a_o = 1$ means that we examine only those time slots containing pulses.

For the general form of θ, D. Slepian and E. N. Gilbert of Bell Telephone Laboratories* have developed an algorithm for determining the pattern that yields the maximum value of θ. Their result is particularly simple when $kQ \ll 1$; then we can approximate x by

$$1 + \sum_1^\infty a_n e^{-(\pi/Q)n}$$

and y by

$$2\pi k \sum_1^\infty a_n n e^{-(\pi/Q)n}.$$

Under this condition Gilbert and Slepian have shown that the pulse

* Private communication.

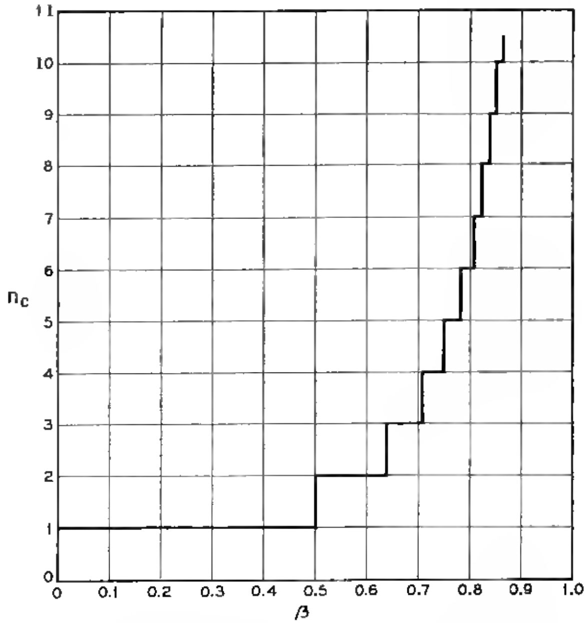

Fig. 1 — n_c vs β for random impulse excitation.

pattern giving the largest value of θ is specified by all pulses present for $n \geqq n_c$ and pulses absent for $n < n_c$. The value of n_c is obtained from*

$$\frac{\beta^{n+1}}{(1 - \beta)^2} = n_c(1 + b) - \frac{a}{2\pi k}. \tag{14}$$

where $\beta = e^{-(\pi/Q)}$. For random impulse excitation $a = 0 = b$. For this case, n_c versus β obtained from (14) is shown in Fig. 1. For $\beta < \frac{1}{2}$, all pulses present ($n_c = 1$) yields the maximum value for θ. In the range $\frac{1}{2} < \beta < 0.639$ the pulse immediately adjacent to the origin is dropped out to obtain $\theta_{\max}$ and so on.

The maximum value attained in a specified interval is achieved for the largest β in the interval and the maximum value is given simply by $2\pi k$ times the n_c defined by the β interval. The β intervals corresponding to constant n_c get smaller and smaller as β approaches one. This is illustrated in Fig. 2, where we have plotted n_c against Q rather than β, showing a continuous approximation to the actual staircase characteristic. We note that for $Q = 100$, $n_c = 80$ and $\theta_{\max} = 2\pi k n_c = 160\pi k$. With $k = 10^{-3}$, $\theta_{\max} = 0.16\pi$ radians.

* See Appendix E for the proof.

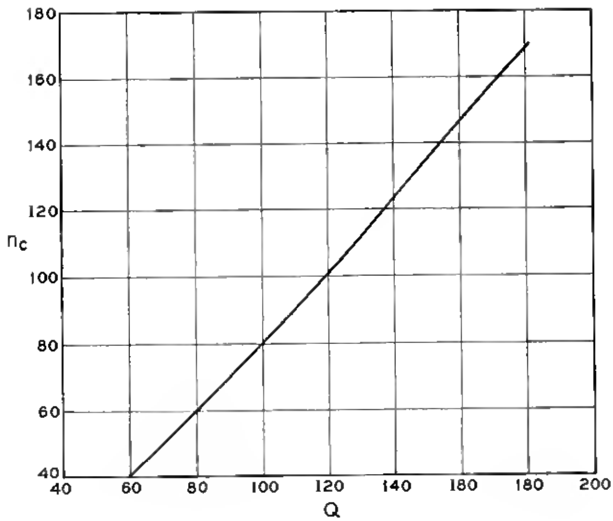

Fig. 2 — n_c vs Q.

For finite width pulses, a and b are non-zero. With raised cosine pulses of pulse width less than 1.5 time slots $a < 0.65$ and $b > -0.75$ with the largest negative value of b corresponding to the consideration of positive going time slots. When the mistuning, k, is positive, the effect of finite pulse width then is to raise the maximum value of n_c over the impulse case and consequently to raise θ_{max}. On the other hand, when $k < 0$, θ_{max} can be reduced over the impulse case. We will demonstrate this effect in connection with the cumulative distribution in Section IX of the paper.

As noted previously, the long string of spaces implied by large n_c make the timing wave amplitude so small as to be useless in a real repeater. The timing wave amplitude can be increased by forcing a periodic pulse pattern. With the constraint that every Mth pulse must occur, the pattern that yields the maximum value for θ is as before where n_c is now given by

$$\frac{\beta^{n_c+1}}{(1-\beta)^2} = -\frac{a}{2\pi k} + n_c\left[1 + b + \beta^M \frac{(1-\beta^{rM})}{(1-\beta^M)}\right] + \frac{M\beta^M(1-\beta^{rM})}{2\pi k(1-\beta^M)^2} - \frac{rM\beta^{(r+1)M}}{2\pi k(1-\beta^M)}, \tag{15}$$

where r is the largest integer less than n_c/M. It can be seen that (15)

reduces to (14) as $M \to \infty$ as expected. Furthermore, since the difference in the last two terms of (15) is positive and the term added to $1 + b$ is also positive, it is apparent that the effect of the periodic pattern is to reduce n_c and consequently $\theta_{\max}$ as expected.

3.4 *Probability Density Function, $p(\theta)$*

With the above preliminaries disposed of, we will proceed to use (13) to develop an approximate expression for $p(\theta)$. To do this we assume that the logarithm of the characteristic function possesses a power series expansion in the neighborhood of $u = 0 = v$. The general form of this series is[10]

$$\log \varphi(u,v) = \sum_{\substack{r=0 \\ r+s\neq 0}}^{\infty} \sum_{s=0}^{\infty} \frac{\lambda_{rs}}{r!s!} (iu)^r (iv)^s \tag{16}$$

where the λ_{rs} are the semi-invariants of the distribution for x_1 and y_1. Since

$$\frac{\partial \varphi}{\partial u} = \varphi \frac{\partial}{\partial u} [\log \varphi],$$

we may write

$$p(\theta) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\partial}{\partial u} [\log \varphi] \bigg|_{u=-\theta v} \exp [\log \varphi] \bigg|_{u=-\theta v} dv. \tag{17}$$

Using (17) and performing the differentiation indicated in the integrand, we get

$$p(\theta) = \frac{d}{d\theta} \left[\frac{i}{2\pi} \int_{-\infty}^{\infty} \exp\left[\sum_{\substack{r=0 \\ r+s\neq 0}}^{\infty} \sum_{s=0}^{\infty} \frac{\lambda_{rs}}{r!s!} (-1)^r \theta^r (iv)^{r+s} \right] \frac{dv}{v} \right]. \tag{18}$$

We now remove terms from the double summation for which $r + s \leq 2$. The remaining terms we treat as u, and expand e^u in a power series retaining only the first two terms ($e^u \sim 1 + u$). In this case $p(\theta)$ becomes approximately

$$p(\theta) \backsim p_o(\theta) + \sum_{\substack{r \\ r+s>2}}^{r+s=6} \sum_{s} \frac{\lambda_{rs}}{r!s!} (-1)^r p_{rs}(\theta), \tag{19}$$

where

$$p_o(\theta) = \frac{d}{d\theta} \left[\frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dv}{v} \exp -iv(\lambda_{10}\theta - \lambda_{01}) - \frac{v^2}{2} (\lambda_{20}\theta^2 - 2\lambda_{11}\theta + \lambda_{02}) \right],$$

or $p_o(\theta) = (d/d\theta)f_o(\theta)$, where $f_o(\theta)$ is defined by comparison with the above.

Similarly,

$$p_{rs}(\theta) = \frac{d}{d\theta}\left[\theta^r \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{dv}{v}(iv)^{r+s} \cdot \exp -iv(\lambda_{10}\theta - \lambda_{01}) - \frac{v^2}{2}(\lambda_{20}\theta^2 - 2\lambda_{11}\theta + \lambda_{02})\right],$$

or

$$p_{rs}(\theta) = \frac{d}{d\theta}f_{rs}(\theta).$$

An upper limit for the double summation in (19) is set in order to make the approximation for $p(\theta)$ consistent with the number of terms used in the power series expansion for e^u. The reason for 6 as an upper limit will become apparent when we discuss the semi-invariants, λ_{rs}, in detail in Section V. Performing the differentiations and integrations indicated in (19) we finally arrive at

$$p(\theta) \sim \frac{1}{\sqrt{2\pi}}\frac{A_2(\theta)}{A_1(\theta)^{\frac{3}{2}}}\exp\left[-\frac{A_o(\theta)^2}{2A_1(\theta)}\right] \cdot\left\{1 + \sum_{r}\sum_{\substack{s \\ r+s>2}}^{r+s=6}(-1)^r\frac{\lambda_{rs}}{r!s!}\theta^r\left[\frac{H_{r+s}\left(\frac{A_o(\theta)}{\sqrt{2A_1(\theta)}}\right)}{(\sqrt{2A_1(\theta)})^{r+s}} + \frac{H_{(r+s)-1}\left(\frac{A_o(\theta)}{\sqrt{2A_1(\theta)}}\right)}{(\sqrt{2A_1(\theta)})^{(r+s)-1}}\cdot\frac{A_{rs}(\theta)}{\theta A_2(\theta)}\right]\right\}, \tag{20}$$

where

$$A_o(\theta) = \lambda_{10}(\theta - \theta_o),$$

$$A_1(\theta) = \lambda_{20}\theta^2 - 2\lambda_{11}\theta + \lambda_{02},$$

$$A_2(\theta) = \lambda_{10}[\theta(\lambda_{20}\theta_o - \lambda_{11}) - (\lambda_{11}\theta_o - \lambda_{02})],$$

$$A_{rs}(\theta) = s\lambda_{20}\theta^2 + (r - s)\lambda_{11}\theta - r\lambda_{02},$$

and

$$\theta_o = \frac{\lambda_{01}}{\lambda_{10}}.$$

The H's are Hermite polynomials defined by

$$H_n(Z) = (-1)^n e^{Z^2} \frac{d^n}{dZ^n} (e^{-Z^2}).$$

The result in (20) gives a general expression for $p(\theta)$ as a function of the semi-invariants of the distribution of x_1 and y_1. The solution obtained is approximate in that it depends upon an asymptotic expansion analogous to the Edgeworth Series. As noted by Cramer,[9] one is not particularly interested in whether series of this type converge or not, but whether a small number of terms suffice to give a good approximation to the probability density function over a specified range of its argument. In our case, the statistical properties of the input pulse pattern, and the parameters of the timing circuit are controlling in this regard. With this in mind, the determination of the range in θ over which a valid approximation may be obtained in various cases is deferred for the present.

IV. CUMULATIVE DISTRIBUTION FUNCTION

The cumulative distribution function $F(\theta)$ may be determined using the results derived in the preceding section. Beginning with (19) we may write

$$p(\theta) \sim f_o'(\theta) + \sum_{\substack{r \\ r+k>2}}^{r+k=6} \sum_{k} \frac{\lambda_{rk}}{r!k!} (-1)^r f_{rk}'(\theta). \tag{21}$$

By definition*

$$F(\theta) = \int_{-\infty}^{\theta} p(u)\, du.$$

Integrating (21) between the limits indicated, $F(\theta)$ becomes

$$F(\theta) \doteq f_o(\theta) + \sum_{\substack{r \\ r+s>2}}^{r+s=6} \sum_{s} \frac{\lambda_{rs}}{r!s!} (-1)^r f_{rs}(\theta) + \frac{1}{2}. \tag{22}$$

Referring back to (19) and performing the integration over v necessary to determine $f_o(\theta)$ and $f_{rk}(\theta)$, we get

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* The significance of the lower limit of integration in the definition of $F(\theta)$ will be discussed in connection with the numerical results.

$$F(\theta) \sim \frac{1}{2} + \frac{erf}{2}\left[\frac{A_o(\theta)}{\sqrt{2A_1(\theta)}}\right] - \frac{1}{\sqrt{2\pi A_1(\theta)}}$$

$$\cdot \exp\left[-\frac{A_o(\theta)^2}{2A_1(\theta)}\right] \sum_{\substack{r \\ r+s>2}}^{r+s=6} \sum_{s} \frac{\lambda_{rs}}{r!s!} (-1)^r \theta^r \cdot \frac{H_{(r+s)-1}\left(\frac{A_o(\theta)}{\sqrt{2A_1(\theta)}}\right)}{(\sqrt{2A_1(\theta)})^{(r+s)-1}}, \tag{23}$$

where $A_o(\theta)$, $A_1(\theta)$ and H_{r+s-1} have been previously defined.

V. SEMI-INVARIANTS FOR THE DISTRIBUTION OF x AND y

In this section we consider the coefficients of the power series expansion for the logarithm of the characteristic function $\varphi(u,v)$. These are determined as functions of the parameters of the timing circuit, and the excitation and provide the necessary information for an explicit solution for $p(\theta)$ and $F(\theta)$. A closed form for the λ_{rs} is obtainable for all excitations of interest under the condition $p = \frac{1}{2}$ (pulses and spaces equally likely). [The semi-invariants for any p can be obtained by appropriate differentiations of log $\varphi(u,v)$. We have not expended the energy for this exercise.] The semi-invariants are shown below for random impulse excitation under the condition $kQ \ll \pi$ and are derived for all excitations we consider in Appendix D.*

$$\lambda_{10} = \frac{1}{2(1-\beta)} \qquad \lambda_{01} = \frac{\pi k\beta}{(1-\beta)^2} \tag{24}$$

$$\lambda_{rs}\,|_{r+s>1} = \frac{(-1)^s B_{r+s}(2^{r+s}-1)}{} \cdot (2\pi k)^s \frac{d^s}{dg^s}\left(\frac{1}{1-e^{-g}}\right) \tag{25}$$

where $\beta = e^{-(\pi/Q)}$, $g = \pi/Q\,(r + s)$, and the B_{r+s} are Bernoulli numbers. Since $B_{r+s} = 0$ for $r + s$ odd and >1, we note that the odd order semi-invariants given in (24) and (25) vanish beyond order 1. Therefore since the λ_{rs} for $r + s = 3$ are zero, one can extend the upper limit in the double summation in (19) to 6, and still maintain consistency with the fact that only 2 terms in the power series expansion for the exponential, e^u, were used in the approximation for $p(\theta)$. This conclusion is valid for all excitations of interest.

VI. BEHAVIOR OF $p(\theta)$ FOR LARGE Q

When the Q of the resonant circuit becomes large, the past history of the input signal becomes increasingly important in determining the

* The more general semi-invariants without the restriction $kQ \ll \pi$ are given in Appendix D; however, they are too long to be repeated here.

statistical properties of x and y. This follows from the form of the exponential term in the expressions for x and y given in (5). Invoking the Central Limit Theorem under this condition, one would expect the values of x and y to begin heaping up about their respective means with the probability density function $p(x,y)$ approaching a two dimensional normal distribution. Analogous behavior is expected of θ and we will now consider $p(\theta)$ as given by (20) in the neighborhood of its mean for large Q. The discussion is restricted to the case of random impulse excitation, but the results for other excitations parallel those of this section.

To determine $p(\theta)$ near its mean, we write, using the previous condition $kQ \ll \pi$,

$$\theta \doteq \frac{y}{x} \doteq \frac{2\pi k \sum_{n=0}^{\infty} a_n n\, e^{-\alpha n}}{\sum_{n=0}^{\infty} a_n\, e^{-\alpha n}}, \tag{26}$$

where

$$\alpha = \frac{\pi}{Q}.$$

For this to hold as Q becomes arbitrarily large, we require the kQ product to be constant. Since

$$x \sim \sum_{n=0}^{\infty} a_n\, e^{-\alpha n},$$

θ can also be written as

$$\theta \sim -2\pi k \frac{d}{d\alpha} [\log x] = -2\pi k \frac{d}{d\alpha} \left[\log \frac{x}{\bar{x}} + \log \bar{x} \right], \tag{27}$$

where $\bar{x}$ is the average value of x. Expanding $\log x/\bar{x}$ in a power series in the neighborhood of 1 (x near $\bar{x}$), and keeping only the first term, θ becomes

$$\theta \sim -2\pi k \frac{d}{d\alpha} [\log \bar{x}] - 2\pi k \frac{d}{d\alpha} \left[\frac{x - \bar{x}}{\bar{x}} \right]. \tag{28}$$

Differentiating the above with respect to α we get for θ in the neighborhood of its mean

$$\theta \sim \frac{\bar{y}}{\bar{x}} + \frac{\bar{x}y - x\bar{y}}{\bar{x}^2}. \tag{29}$$

In determining this result we make use of the fact that

$$\bar{y} = -2\pi k \frac{d}{d\alpha} [\bar{x}]. \tag{30}$$

Using (29) one can determine the logarithm of the characteristic function of θ, and the associated semi-invariants of the θ distribution. When this is done, the mean of θ is

$$\bar{\theta} \sim \frac{\bar{y}}{\bar{x}} = \theta_o = \frac{2\pi k\beta}{1 - \beta} \tag{31}$$

which also can be derived directly from (29). The standard deviation and the 4th semi-invariant are given by

$$\sigma = \sqrt{\frac{2(2\pi k)^2\beta^2}{(1 - \beta)(1 + \beta)^3}}$$

$$\begin{aligned}\lambda_4 = \frac{-2(2\pi k)^4\beta^4}{(1 - \beta^4)} \Bigg[1 &- \frac{4\beta^3(1 - \beta)}{(1 - \beta^4)} + \frac{6\beta^2(1 + \beta^4)(1 - \beta)^2}{(1 - \beta^4)^2} \\ &- \frac{4\beta(1 - \beta)^3(1 + 4\beta^4 + \beta^8)}{(1 - \beta^4)^3} \\ &+ \frac{(1 - \beta)^4(1 + 11\beta^4 + 11\beta^8 + \beta^{12})}{(1 - \beta^4)^4} \Bigg]\end{aligned} \tag{32}$$

with $\beta = e^{-\alpha}$. These same results can be derived using (20) and including only the first correction term from the double sum (i.e., only those λ_{rs} for which $r + s = 4$). The details of the calculation along with the λ_{rs} of interest are given in Appendix D. The final result for $p(\theta)$ is

$$p(\theta) \sim \frac{1}{\sqrt{2\pi}\,\sigma} \exp - \frac{(\theta - \theta_o)^2}{2\sigma^2} \left(1 + \frac{\lambda_4}{4!} \frac{H_4 \dfrac{\theta - \theta_o}{\sqrt{2}\,\sigma}}{4\sigma^4} \right). \tag{33}$$

The above equation for $p(\theta)$ is in the form of the standard Edgeworth approximation. In the limit as Q becomes large ($\beta \to 1$), and with kQ constant, $p(\theta)$ reduces to

$$p(\theta) \sim \frac{1}{\sqrt{2\pi\sigma}} \exp - \frac{(\theta - \theta_o)^2}{2\sigma^2} \left[1 - \frac{5\pi}{128Q} H_4 \left(\frac{\theta - \theta_o}{\sqrt{2}\sigma} \right) \right] \tag{34}$$

with $\theta_o \doteq 2kQ$ and $\sigma \doteq k\sqrt{\pi Q}$. Equation (26) indicates the approach to the normal law as Q becomes large with the first correction term going as $1/Q$. The above results for θ_o and σ correspond to those derived

earlier by Bennett[1] by another method. If we rewrite σ as $kQ\sqrt{\pi/Q}$ we notice that $p(\theta)$ becomes more peaked with increasing Q, and falls off quite rapidly as θ departs from the mean. In the high Q case the concentration about θ_o becomes more pronounced as expected.

It is to be emphasized that the general properties of $p(\theta)$ for large Q demonstrated here will be true for the other inputs also. For example, with random impulse excitation plus 1 out of M pulses forced, the average value will remain the same as above but σ will be a function of M;

$$\sigma \doteq kQ\sqrt{\frac{\pi}{Q}\frac{M(M-1)}{(M+1)^2}} \qquad \text{for } \frac{Q}{M\pi} \gg 1.$$

The effect of M is to reduce σ and therefore increase the concentration about the mean. As M becomes large (fewer pulses required to occur), the effect of M becomes insignificant for this large Q case.

VII. NUMERICAL RESULTS FOR $p(\theta)$ AND $1 - F(\theta)$: IMPULSE EXCITATION

7.1 $p(\theta)$

To determine the behavior of the probability density function for finite Q, we must use the general form of the approximation to $p(\theta)$ given by (20), since most of the approximations made in the previous section for Q arbitrarily large are no longer valid. By way of illustration we consider the case $Q = 100$, $k = 10^{-3}$ with impulse excitation and all pulses random ($p = \frac{1}{2}$). For negative mistuning, $k = -10^{-3}$, the curve for $p(\theta)$ will be identical with that for k positive except that θ is replaced with $-\theta$. The result for the probability density function is shown in Fig. 3. The calculations* upon which this curve is based include the first and second correction terms of (20); i.e., terms for which $r + s = 4$ and $r + s = 6$. Points beyond $\theta = 0.13$ radians on the lower end and $\theta = 0.35$ radians on the upper end are not included, since the approximation begins to fail at these extremes. More specifically, the probability density obtained from (20) goes negative somewhere between $\theta = 0.13$ radians and $\theta = 0.12$ radians and $\theta = 0.35$ and $\theta = 0.36$ radians. However, as we shall see later, up to these points the results for the cumulative distribution are in good agreement with computer simulation. The cumulative distribution is also shown on Fig. 3 to point out the fact that the median occurs slightly below the approximate mean given by $2kQ$. In addition, it is apparent from the shape of $p(\theta)$ and

* Equation (20) and all subsequent calculations for $p(\theta)$ and $F(\theta)$ were programmed for the IBM 7090 computer by Miss E. G. Cheatham.

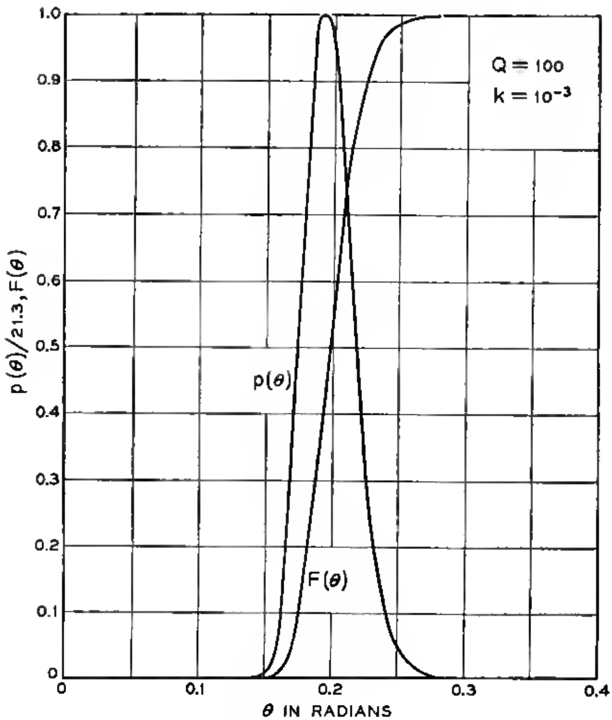

Fig. 3 — $p(\theta)$ and $F(\theta)$ as a function of θ for $k = 10^{-3}$ and $Q = 100$. Random impulse excitation.

$F(\theta)$ that the probability density is skewed in the direction of increasing phase error. This is more easily visualized from Fig. 4 where we have shown $p(\theta)$ as in Fig. 3 plotted on log paper. The normal probability density with the same mean and variance as our computed curve is also shown to further illustrate the skewness.

On Fig. 5 we have plotted $p(\theta)$, as defined in (20), to illustrate the contribution of its constituent terms. From this figure we see that the principal term (always positive) predominates over most of the range. At the tails, the terms involving λ_{rs} for $r + s = 4$ pulls $p(\theta)$ in and forces the density to become negative. The last term in the approximation, for which $r + s = 6$, serves to extend the region over which $p(\theta)$ remains positive.

When $1/M$ pulses are forced, the skewness is reduced, as is the variance. There are several ways of explaining this effect. First, as discussed in Section 3, the denominator of θ in (8) or (9) is raised, thereby reducing

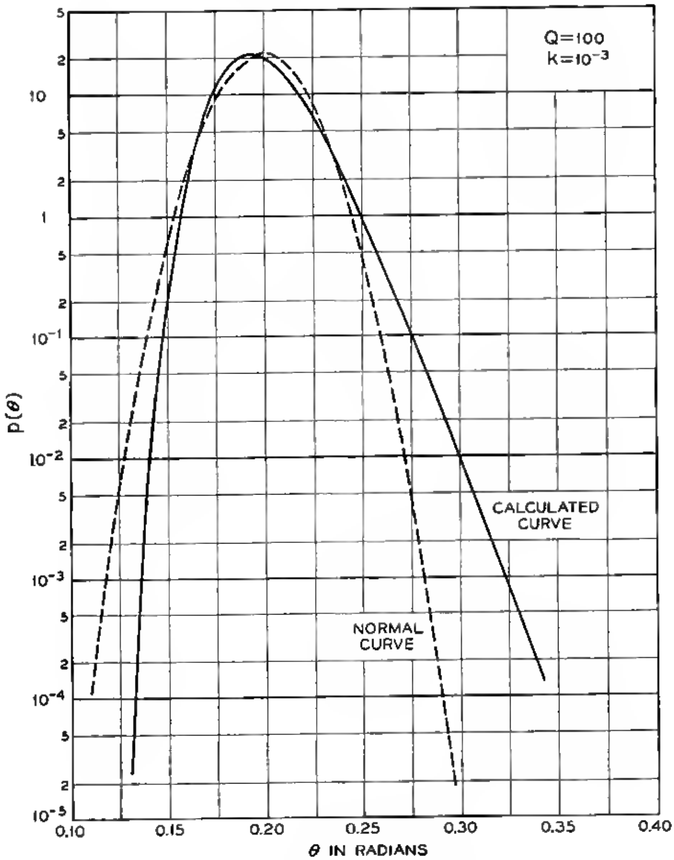

Fig. 4 — $p(\theta)$ for $k = 10^{-3}$ and $Q = 100$. The normal curve with the same mean and variance is also shown for comparison. Random impulse excitation.

the range of variation of the timing wave amplitude and confining θ to a narrower range. This is expected from the physical standpoint, since forcing a periodic pattern with the remaining pulses and spaces equally likely is similar to increasing the probability of occurrence of a pulse in an all-random sequence. Since the pulses, when they occur, have the proper spacing, they will tend to correct for the departure of the zero crossings from the mean that has occurred during the free response of the tuned circuit in the absence of a pulse. Indeed, in the limit when $M = 1$ (all pulses definitely occur), all the probability is concentrated at the mean, $2kQ$, which is identical to the steady state phase shift of

the tuned circuit in response to a sine wave at the pulse repetition frequency. This behavior is also predicted mathematically from (20) and the fact that λ_{rs} goes to zero for $r + s > 1$ when $M = 1$. The same effect occurs when Q approaches infinity with kQ constant and it can be shown from the results of the previous section that $p(\theta)$ goes to $\delta(\theta)$ when the limit is taken. In this light, we can view the introduction of forced pulses as effectively increasing the Q of the tuned circuit while maintaining kQ fixed.

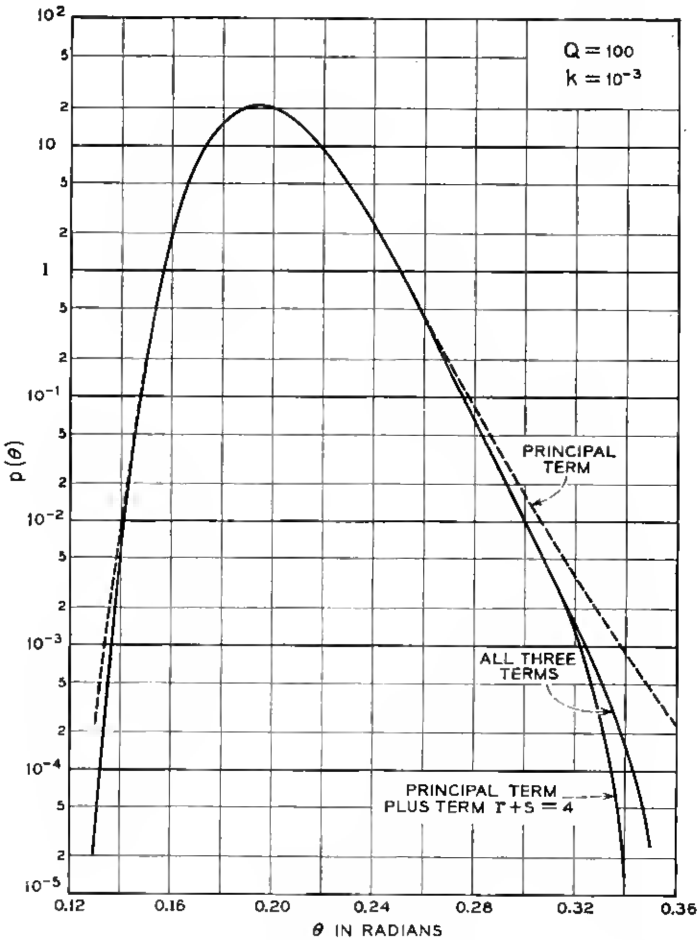

Fig. 5 — Contributions of various terms involved in the $p(\theta)$ approximation given by (20). Random impulse excitation is assumed, with $k = 10^{-3}$ and $Q = 100$.

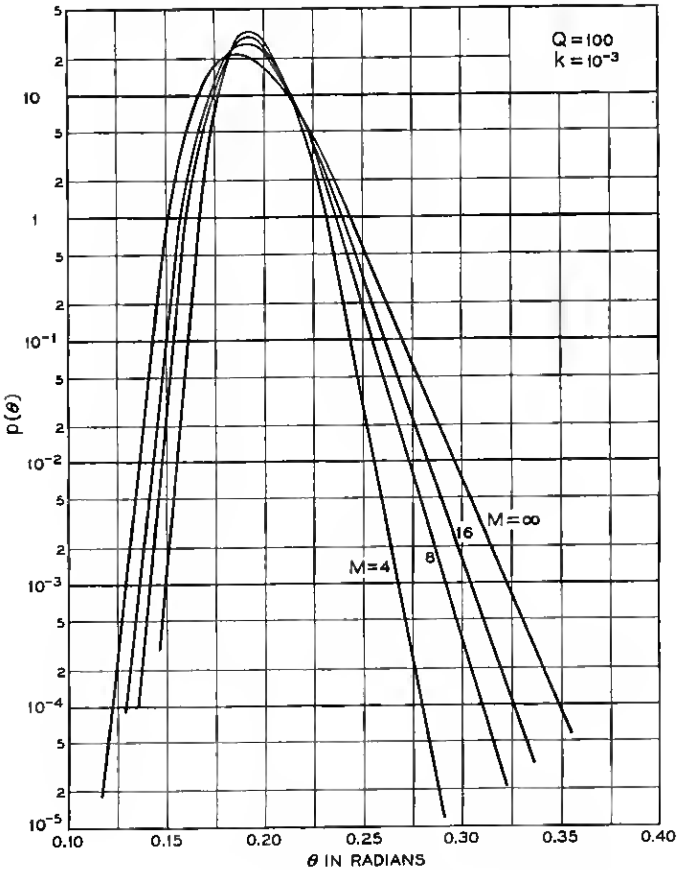

Fig. 6 — The effect on $p(\theta)$ of requiring $1/M$ impulses to occur. $k = 10^{-3}$, $Q = 100$.

In practical applications, the effect of a pulse at the origin is of particular interest. Mathematically, this corresponds to $M = \infty$. Physically this means we examine and record phase error only for those time slots containing a pulse. Fig. 6 illustrates the narrowing of the density function for $M = \infty$ (pulse at the origin), and $M = 16$, 8, and 4. It is interesting to note that, for these cases, the probability density function remains positive over the range of θ we have used in the computations from 0.1 to 0.4 radian. This encompasses values of $p(\theta) < 10^{-7}$ on the left of the mean and $p(\theta) < 10^{-5}$ to the right of the mean. This is to be expected since λ_{rs} decrease with decreasing M for $r + s \geqq 2$, thereby

reducing the importance of the terms involving the Hermite polynomials in (20) and improving the approximation.

Fig. 7 depicts the behavior of $p(\theta)$ as Q grows with kQ fixed at 0.1. The results are consistent with the predictions of the previous section.

7.2 $1 - F(\theta)$

For a closer inspection of the behavior of the distribution at its tails, $1 - F(\theta)$ will be examined. This function as evaluated from (23) for

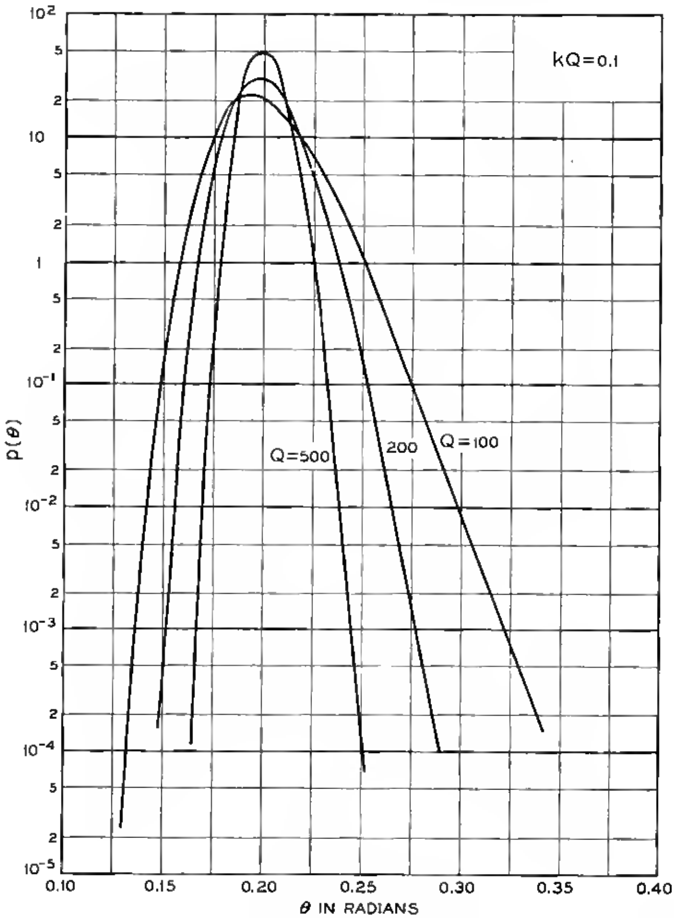

Fig. 7 — The effect on $p(\theta)$ of increasing Q with $kQ = 0.1$ and random impulse excitation.

$Q = 100$, $k = 10^{-3}$, and purely random excitation ($p = \frac{1}{2}$) is shown in Fig. 8. The plot shown gives the probability that θ deviates from its mean by more than some constant C times σ. In the same figure a comparison of the calculated approximation with the normal curve of identical mean and standard deviation indicates a substantial departure from the normal law as the phase error increases. When periodic patterns are interspersed with the random train, the departure from the mean is further reduced, as can be seen from Fig. 9. Similar behavior is exhibited in Fig. 10, where Q is increased from 100 to 500 and kQ maintained constant at 0.1.

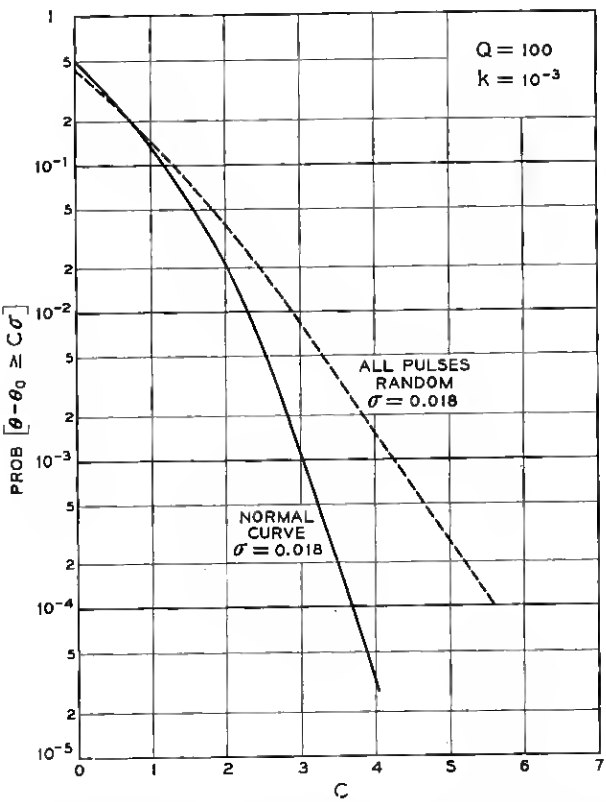

Fig. 8 — Comparison of $1 - F(\theta)$ with the normal curve in the vicinity of the tails. The normal curve is computed assuming the same mean and variance used in determining $1 - F(\theta)$. Random impulse excitation with $Q = 100$ and $k = 10^{-3}$ is assumed for computing $1 - F(\theta)$.

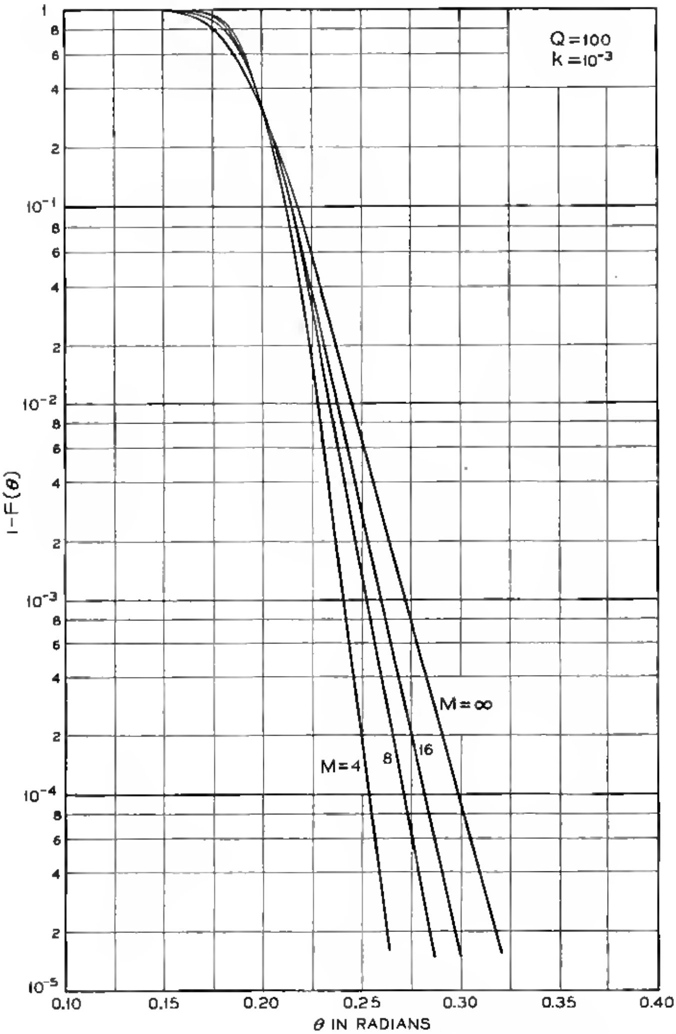

Fig. 9 — The effect on $1 - F(\theta)$ of requiring $1/M$ impulses to occur. $k = 10^{-3}$, $Q = 100$.

7.3 *Comparison with other approaches*

Since we have made approximations in arriving at our expression for the phase error, it is natural to ask how these approximations affect our computed results. A comparison of our results with two other approaches

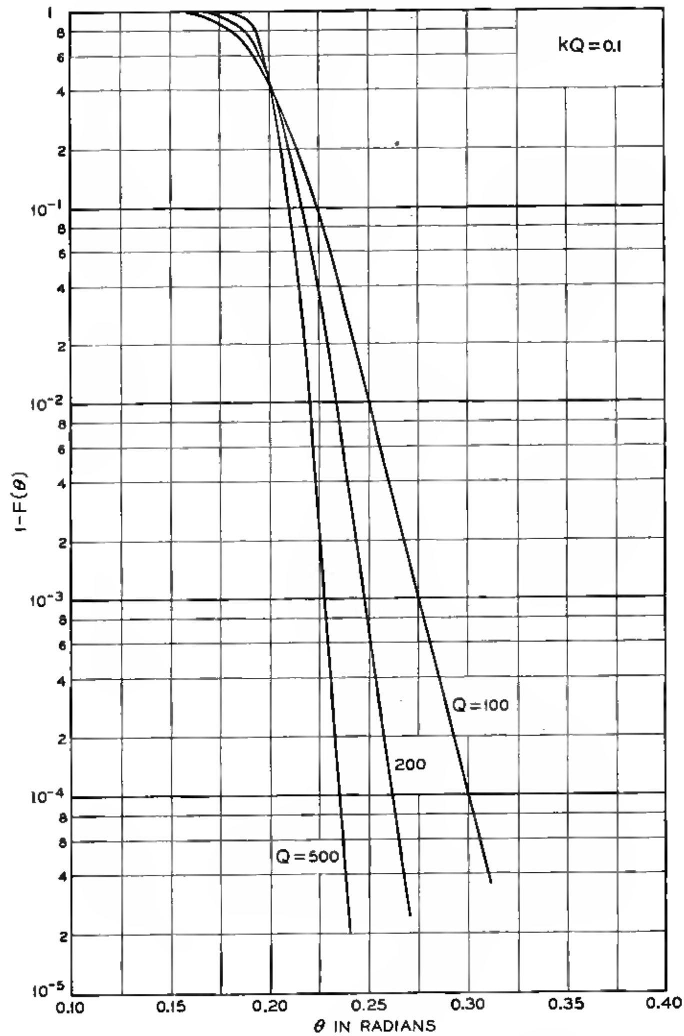

Fig. 10 — The effect on $1 - F(\theta)$ of increasing Q with $kQ = 0.1$ and random impulse excitation.

will be made for the case of impulse excitation. We recall from Section 2 that the phase error under impulse excitation is given by

$$\tan \theta = \frac{y}{x}.$$

For kQ sufficiently small we can write

$$\frac{\theta}{2\pi k} \doteq \frac{\sum_{n=0}^{\infty} n a_n \beta^n}{\sum_{n=0}^{\infty} a_n \beta^n}. \tag{35}$$

The approximation of tan θ by its argument is not crucial in this case, since a straightforward transformation can be made on the probability distribution to correct for this approximation [i.e., $p(\theta) = \sec^2 \theta p(\tan \theta)$].

H. Martens* shows that (35) can be manipulated to yield a recursion relationship for the phase error that is in a convenient form for digital computer evaluation. T. V. Crater and S. O. Rice used this approach in some of their work, and a probability distribution so determined is shown by the dots in Fig. 11 for $Q = 125$. For the same value of Q, we have computed the probability distribution from the series in (23), and it is displayed as the solid curve of Fig. 11. It can be seen that the agreement between the two approaches is excellent. The scattering of the "experimental" points at the 10^{-3} level and below is due to the limited number of pulse positions considered by Crater and Rice. Specifically, 10^4 pulse positions were processed after an initial transient of some 5×10^3 pulse positions had elapsed.

In addition, S. O. Rice in unpublished work has shown that the tail of the distribution should behave as $A(\frac{1}{2})^{\theta/2\pi k}$, where A is an unknown constant. When we take the values of θ at the 10^{-3} and 10^{-4} levels and substitute these in Rice's asymptotic form and form a ratio, the constant A cancels out and we should obtain 10. The actual value for the ratio is 10.9, which tends to indicate that the asymptotic behavior has virtually been reached. This suggests that an extrapolation of the distribution to larger values of θ by merely continuing with the same slope should be valid.

We also note that we can write

$$(\tfrac{1}{2})^{\theta/2\pi k} = (\tfrac{1}{2})^{\theta Q/\pi \theta_0}$$

where we have made use of $\theta_0 = 2kQ$. With kQ constant, one would expect the cumulative probability to fall off faster for larger Q, as is indeed the case. The slopes of the curves of Fig. 10 follow Rice's predictions quite closely.

While the above comparisons are comforting, they only indicate that our final expressions for $p(\theta)$ and $F(\theta)$ are accurate for computing these quantities from the initial defining equation for θ. Approximations have been made in arriving at the starting relationship. A check on these initial approximations may be obtained from a simulation of the problem.

* Unpublished memorandum.

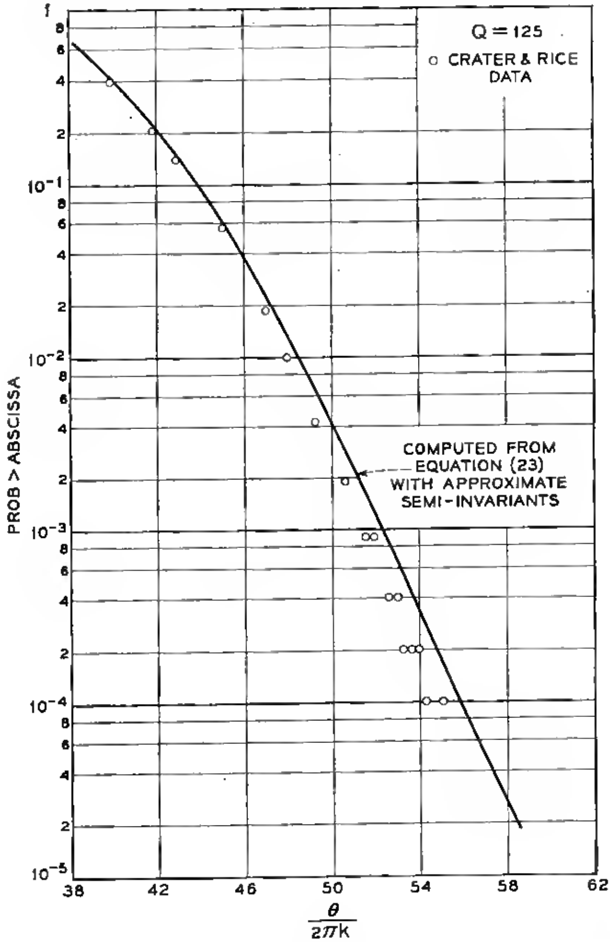

Fig. 11 — Comparison of $1 - F(\theta)$ computed by (23) with the results of the Crater-Rice simulation for $Q = 125$. Random impulse excitation is assumed.

One such simulation has been accomplished by Miss M. R. Branower using a combination of analogue and digital computers. The principal errors introduced in this process involve the stability of the analogue computer with time and the number of pulses processed. For a tuned circuit characterized by a Q of 125 and mistuning $k = +10^{-3}$, the computer simulation yields the results of Fig. 12. Results obtained using (23), the exact semi-invariants of Appendix C, and the tan θ transformation mentioned previously yield the "computed curve" of Fig. 12.

Again the results are in very close agreement. To indicate the effect of the approximation $kQ \ll \pi$, we have repeated the computed curve of Fig. 11 on Fig. 12.

VIII. RAISED COSINE EXCITATION

8.1 *Results for $1 - F(\theta)$*

With raised cosine excitation, the computations are performed as before and only the semi-invariants λ_{rs} for $r + s = 1$ are changed from the

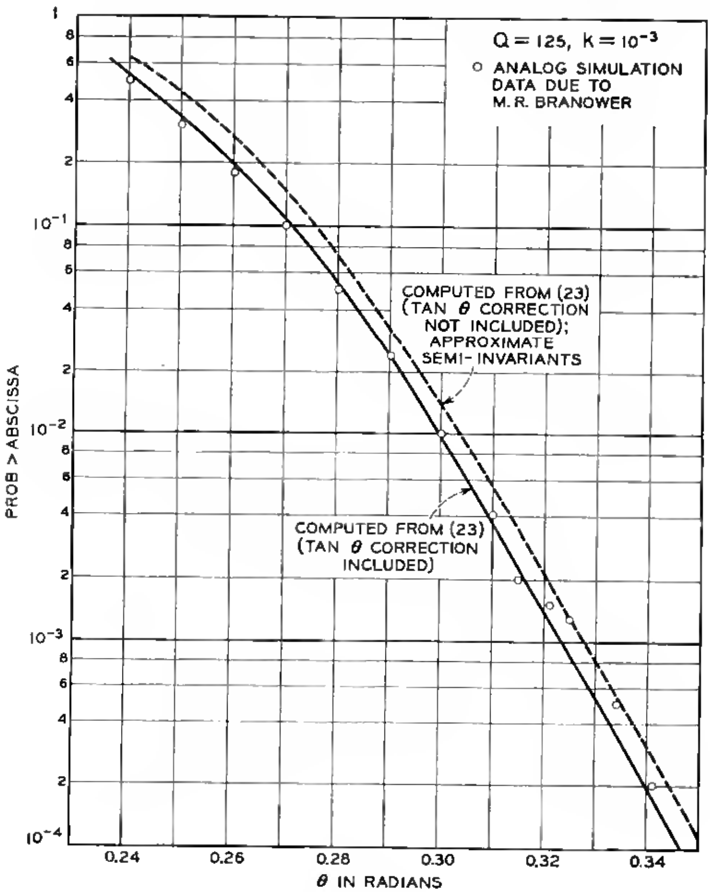

Fig. 12 — Comparison of $1 - F(\theta)$ computed by (23) with the results of an analog simulation due to M. R. Branower. Random impulse excitation with $Q = 125$ and $k = 10^{-3}$ is assumed. The effect of the tan θ approximation is shown together with results for both approximate and exact semi-variants.

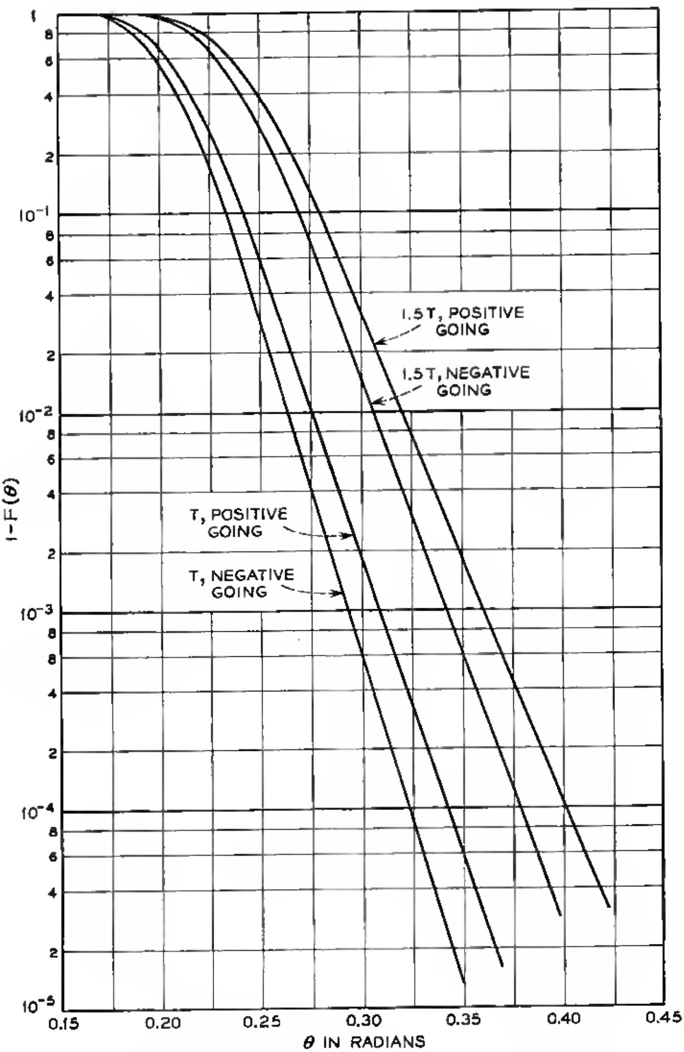

Fig. 13 — Plot of $1 - F(\theta)$ for raised cosine excitation. Pulses of width T and $1.5T$ are assumed in the calculation. The distribution of the phase error for both positive and negative-going zero crossings is shown. $Q = 100$, $k = 10^{-3}$.

previous case. Results obtained for this excitation are shown on Fig. 13, where it is apparent that the use of widest pulses and positive-going zero crossings yields the largest phase error. The effect of Q and M with this type of input is the same as with impulses.

8.2 *Comparison with another approach when $k = 0$*

In the absence of mistuning, the phase error becomes

$$\theta = \frac{a}{x + b}, \tag{36}$$

and the probability distribution for θ may be obtained by methods given previously, or by the following relationship:

$$\begin{aligned} \text{Prob}\,(\theta \geqq \lambda) &= \text{Prob}\left(\frac{a}{x + b} \geqq \lambda\right) \\ &= \text{Prob}\left(x \leqq \frac{a - b\lambda}{\lambda}\right). \end{aligned} \tag{37}$$

Therefore, if the distribution for x is known, the distribution for θ may be determined from it. The random variable x is the normalized timing wave amplitude defined by Rowe. This random variable has been considered by S. O. Rice in unpublished work and he has developed a procedure for closely approximating its probability distribution. Using the method of moments, one of the authors also computed this distribution. The results were in excellent agreement with Rice's results and the cumulative distribution obtained by the moment method is shown in Fig. 14. It can be shown that the probability density for x is unimodal and symmetric about its mean; therefore, the data on Fig. 14 suffices to specify the complete distribution. With this data and (37) we can determine the distribution for θ. Alternately, we can use (23) to make this computation. A comparison of the distribution obtained by the two approaches is shown in Fig. 15 and it can be seen that the agreement is very close. Thus we have found another check on our series approximation for $p(\theta)$. Conversely, we can use the distribution for θ to compute the distribution for x. In this regard it is interesting to note that when the Edgeworth expansion including semi-invariants through order 6 is used to approximate the distribution for x, the density function begins to turn negative in the neighborhood of 3σ from the mean indicating failure of the approximation. On the other hand, using the same number of semi-invariants in the expansion for $p(\theta)$, where θ in this case is essentially the reciprocal of x, we obtain a good approximation to the cumulative distribution for x. This is believed to be due to the narrowness of the range of θ as compared with x; i.e., x varies from 1 to $1/(1 - \beta) = Q/\pi$, while $1/x$ goes from $1 - \beta \doteq \pi/Q$ to 1.

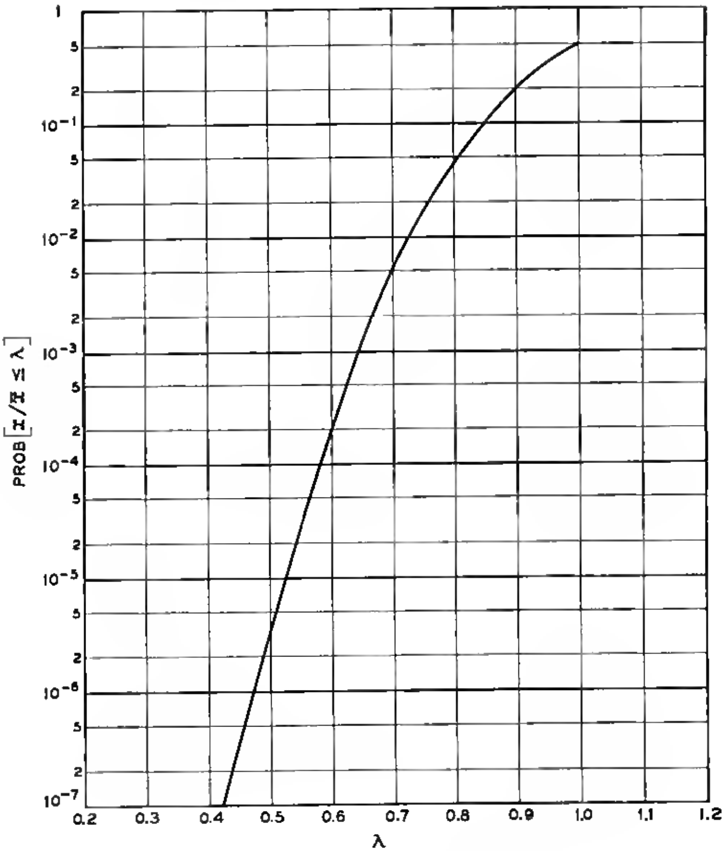

Fig. 14 — Probability distribution of the timing wave amplitude. $Q = 100$.

IX. OPTIMUM TUNING — FINITE PULSE WIDTH

In the case of impulse excitation it should be apparent that zero mistuning, $k = 0$, is the desired objective for no phase error. On the other hand, with finite width pulses zero mistuning does not yield zero phase error. Mistuning can be purposely introduced in the finite pulse width case to make the mean value of θ zero, to minimize the variance of θ, or to optimize some other parameter of the θ distribution.

An approximation to making the mean of θ zero may be obtained by choosing k such that the average value of the numerator of θ is zero. This means that

$$\bar{y}_1 = a + \bar{y} \doteq a + \frac{\pi k \beta}{(1 - \beta)^2} = 0, \tag{38}$$

or

$$k = -\frac{a(1 - \beta)^2}{\pi \beta}. \tag{39}$$

For example, when $Q = 100$ and $a = 0.65$, as for raised cosine pulses of width $1.5T$, then $k = -2.05 \times 10^{-4}$ to satisfy (39). In the high Q case (39) becomes $k \doteq -(a\pi/Q^2)$.

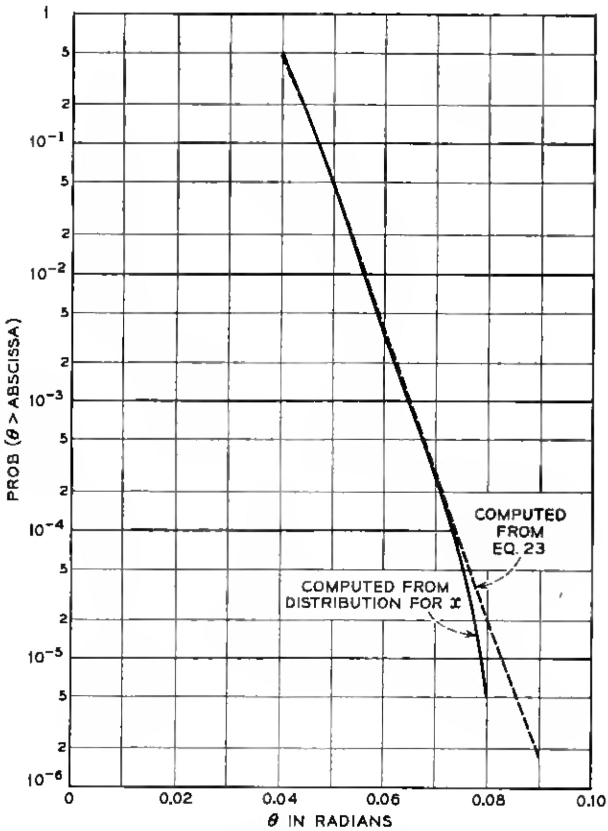

Fig. 15 — Comparison of the distribution of θ as computed by (23) and that determined from the distribution of the timing wave amplitude of Fig. 14. Raised cosine pulses of width 1.5T drive a tuned circuit with a $Q = 100$ and zero mistuning. Timing deviations in the neighborhood of negative-going zero crossings are considered.

When the objective is to minimize the variance of θ, we consider σ as defined in Appendix D; i.e.

$$\sigma = \left| \frac{(\lambda_{20}\theta_o^2 - 2\lambda_{11}\theta_o + \lambda_{02})^{1/2}}{\lambda_{10}} \right|. \tag{40}$$

A plot of σ versus k is shown in Fig. 16, where it is seen that the minimum σ occurs close to the "zero mean" value of k. Probability distributions for values of k that encompass the optimum are shown on Fig. 17. The narrowing of the density function for the optimum value of k is evident.

The results of this section suggest that when the tuned circuit in a self-timed repeater is adjusted, it should be excited with a random pulse train and the tuning adjusted to minimize the jitter on the leading edge

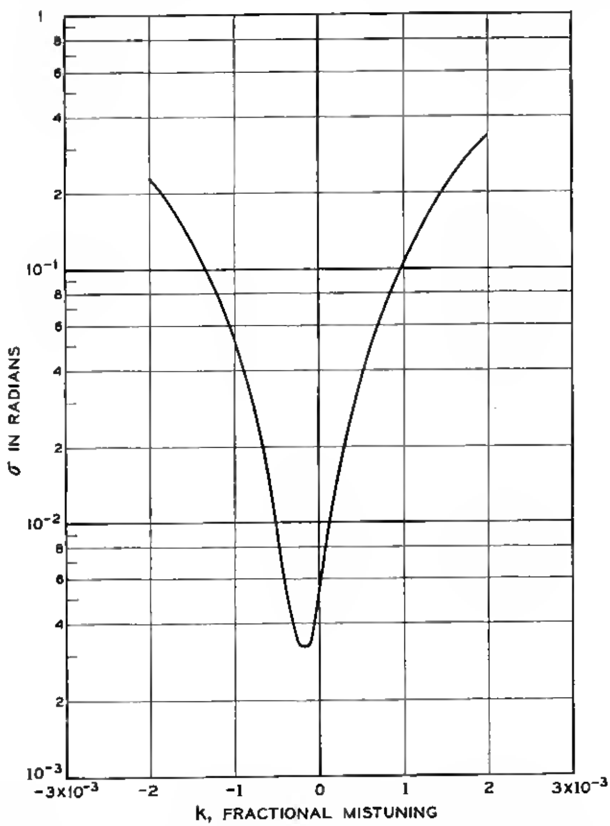

Fig. 16 — Standard deviation of phase error as a function of mistuning with raised cosine pulses 1.5T wide. Negative-going zero crossings are considered. Q = 100.

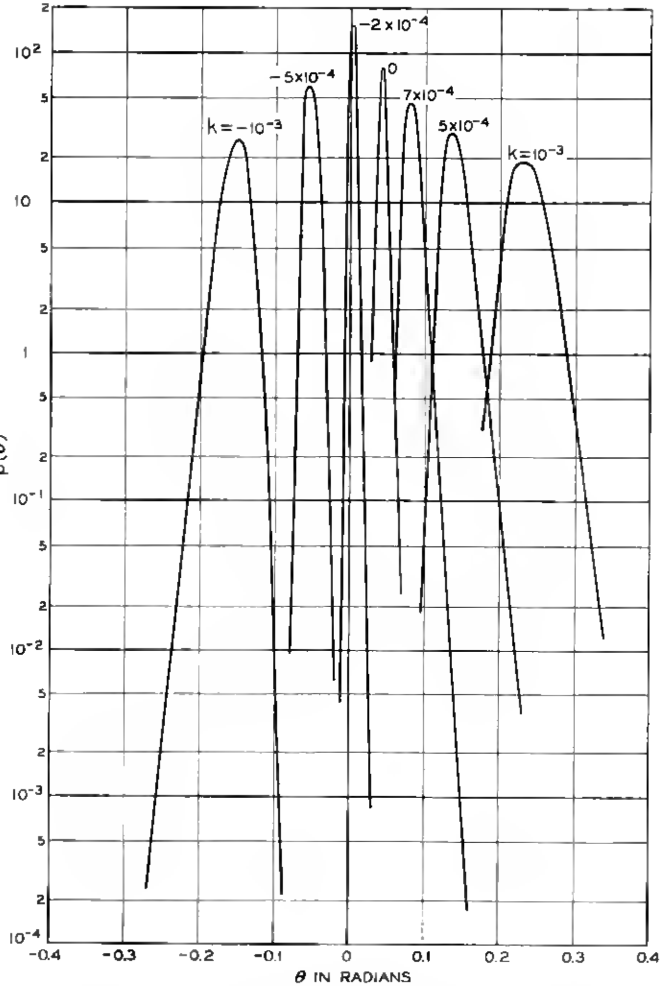

Fig. 17 — $p(\theta)$ for raised cosine excitation with various mistunings in the neighborhood of the optimum mistuning. Negative-going zero crossings and pulses $1.5T$ wide are assumed in making the calculations. $Q = 100$.

of the output pulse train as viewed, for example, on an oscilloscope. This is the method used for the adjustment of the repeater of Ref. 8.

X. PARTIAL RETIMING

In Section VIII we have shown that, in the absence of mistuning, the variable θ can be related to the normalized timing wave amplitude x

and the distribution for θ determined from the distribution for x. Here we will also make use of the distribution for x in order to analyze an idealized version of a forward-acting partial retiming scheme. The scheme we consider has been described by E. D. Sunde[5] and analyzed for periodic pulse patterns in Ref. 7. We make the same assumptions here as in the later reference, namely

1. The pulses exciting the tuned circuit are so narrow that they can be considered impulses. They are obtained by processing incoming pulses to the repeater and they excite a simple tuned circuit.
2. The timing wave is so clamped that its maximum excursion is at ground.
3. Reconstruction of the raised cosine pulse takes place when the algebraic sum of the timing wave and the raised cosine pulse crosses a threshold assumed to be at half the peak pulse amplitude.

For random impulse excitation of the tuned circuit prior to $t = 0$ and the definite occurrence of a pulse at $t = 0$, we have, according to the above assumptions (with no pulse overlap)

$$\frac{1}{2}\left(1 + \cos\frac{2\pi ts}{T}\right) - \frac{x}{2\bar{x}}\left(1 - \cos\frac{2\pi t}{T}\right) = \frac{1}{2} \tag{41}$$

for $|t| \leqq T/2s$
where

$$x = \sum_{n=0}^{\infty} a_n\beta^n,$$

$$a_o = 1 \text{ (the pulse at the origin definitely occurs)},$$

and

$$\bar{x} \equiv \text{average value of } x.$$

Equation (41) is based on the assumption that the average timing wave has a peak-to-peak amplitude equal to the peak pulse height (i.e., when $x = \bar{x}$, the timing wave amplitude varies between -1 and 0). If we define t_p as the time at which regeneration takes place and $\theta_p = 2\pi t_p/T$ as the corresponding phase angle, then it can be seen from (41) that this phase is a random variable dependent upon the random variable x. We will solve for θ_p under the condition $s = 1$, which means that the information-bearing pulses are resolved.* Under this condition $-(\pi/2) < \theta_p < 0$. Consistent with our previous definition of phase error, we will consider the negative of θ_p, since this makes the phase error positive

* Other pulse widths and different ratios of average timing wave amplitude to pulse peak can be handled, but we will not consider them here.

when we take our reference as the phase corresponding to the time at which the pulse peak occurs (at $t = 0$). In this way a positive phase error corresponds to regeneration prior to the pulse peak and permits direct comparison with the results of section 8 for the complete retiming approach. Solving (41) for $\cos \theta_p$ gives

$$\cos \theta_p = \frac{\frac{x}{\bar{x}}}{1 + \frac{x}{\bar{x}}} \tag{42}$$

and

$$\begin{aligned} \text{Prob}\,(\cos\theta_p \leqq \lambda) &= \text{Prob}\,(\theta_p \geqq \cos^{-1}\lambda) \\ &= \text{Prob}\left(\frac{\frac{x}{\bar{x}}}{1 + \frac{x}{\bar{x}}} \leqq \lambda\right) = \text{Prob}\left(x \leqq \frac{\lambda \bar{x}}{(1 - \lambda)}\right). \end{aligned} \tag{43}$$

It is apparent from the above that we can use the distribution for x to determine the distribution for θ_p. For $Q = 100$, the distribution for x is shown in Fig. 14 and with (43) enables us to obtain the distribution for θ_p as shown in Fig. 18. When we compare this result with that of Fig. 15, which shows $1 - F(\theta)$ for the case of complete retiming, it is apparent that partial retiming results in a considerably larger variation of phase error. This supports the contention made in Ref. 7.

XI. CONCLUSIONS AND FUTURE WORK

We have derived an approximate relationship for the probability density and cumulative distribution for the phase error at the output of a tuned circuit when it is excited by a random or random plus periodic pulse train. The effects of mistuning of the tuned circuit and the finite widths of the driving pulses have been considered. Three independent checks of our results indicate that the expressions given are excellent approximations to the true state of affairs for $kQ < 0.1$ and $Q > 100$. Regions defined by these limits encompass values of k and Q of interest in PCM systems under consideration.

More specifically, we have shown that the distributions are not normal and are skewed in the direction of increasing phase error. When we consider pulse positions in which a pulse definitely occurs, it has been shown that the maximum phase error is bounded. In addition with raised cosine excitation we have demonstrated that the mistuning can be adjusted to minimize the mean or variance of the distribution for the

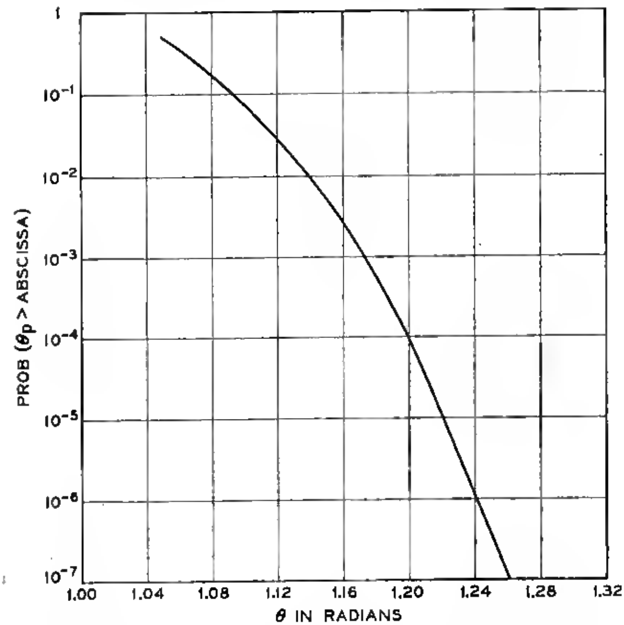

Fig. 18 — Distribution of the phase error with partial retiming. $Q = 100$ and $k = 0$. Raised cosine excitation pulse width $= T$.

phase error. The performance of an idealized version of a forward-acting partial retiming scheme has been analyzed and shown to be considerably inferior to a completely retimed repeater.

There are several desirable directions to proceed from our present position. First, it appears to be possible, in the case where we examine pulses only, to start from the maximum value of θ and work back toward the mean to better approximate the distribution near the tails. S. O. Rice has used this approach in related problems with success. Second, it is of interest to determine the pattern to give the maximum phase error at the output of a string of repeaters. This is not necessarily the pattern that creates θ_{max} in a single repeater. In this regard, we have concentrated on only a single repeater. Obviously it is of interest to extend our results to a repeater string. This extension remains elusive.

XII. ACKNOWLEDGMENTS

The authors are indebted to several people for contributions to this effort. We would like to acknowledge the work of Miss E. G. Cheatham

in computer programming. T. V. Crater kindly made the results of his digital computations available prior to publication. Miss M. R. Branower was most cooperative in providing us with distributions obtained from an analogue simulation of the problem. Our thanks go to E. N. Gilbert and D. Slepian for deriving the conditions for the maximum value of θ. We are grateful to S. O. Rice for helpful discussions, for results on the asymptotic behavior of the probability distribution, and for prior work on a related problem which provided us with helpful clues on how to proceed in our problem.

APPENDIX A. DERIVATION OF EQUATION FOR NORMALIZED TIMING ERROR

A-1. *Response of tuned circuit to random pulse train*

The impulse response of a parallel resonant circuit is well known to be

$$h(t) = \text{Real part of} \left[\frac{1}{C} \left(1 + \frac{j}{2Q} \right) e^{-(\pi/Q) f_o t} \, e^{+j2\pi f_o t} \right]. \tag{44}$$

Following Rowe,[2] we will imply the real part in all subsequent calculations involving complex quantities. The pulse train applied to the tuned circuit is given by

$$r(t) = \sum_{-\infty}^{\infty} a_n g(t - nT), \tag{45}$$

where:

$$a_n = 1 \text{ with probability } p,$$

$$a_n = 0 \text{ with probability } 1 - p, \text{ and}$$

$$g(t) = \text{pulse shape representing the binary 1.}$$

The response of the tuned circuit to $r(t)$ is

$$z(t) = \int_{-\infty}^{t} r(\tau) h(t - \tau) \, d\tau. \tag{46}$$

In view of (45), this can be written

$$z(t) = T \sum_{-\infty}^{+\infty} a_n h(t - nT) \cdot \int_{-\infty}^{(t/T)-n} g(xT) \exp\left[\left(\frac{f_o T \pi}{Q} - j2\pi f_o T \right) x \right] dx. \tag{47}$$

Define

$$f_o \equiv \frac{1+k}{T} = f_r(1+k), \tag{48}$$

with $k \equiv$ fractional mistuning from the pulse repetition frequency. Equation (47) can be manipulated to yield

$$z(t) = |A(t)|\, e^{j[2\pi f_r t + \Phi(t)]}, \tag{49}$$

where

$$\Phi(t) = \tan^{-1}\frac{1}{2Q} + 2\pi f_r k t + \tan^{-1}\frac{\displaystyle\sum_{-\infty}^{\infty} a_n e^{\pi/Q(1+k)n}\left[-I_1\left(\frac{t}{T}-n\right)\sin 2\pi k n + I_2\left(\frac{t}{T}-n\right)\cos 2\pi k n\right]}{\displaystyle\sum_{-\infty}^{\infty} a_n e^{\pi/Q(1+k)n}\left[I_1\left(\frac{t}{T}-n\right)\cos 2\pi k n + I_2\left(\frac{t}{T}-n\right)\sin 2\pi k n\right]} \tag{50}$$

and

$$\begin{aligned} I_1\left(\frac{t}{T}-n\right) &= \mathrm{Re}\int_{-\infty}^{(t/T)-n} g(xT)\exp\left[\left(\frac{\pi}{Q}f_oT - j2\pi f_oT\right)x\right]dx, \quad \text{and} \\ I_2\left(\frac{t}{T}-n\right) &= \mathrm{Im}\int_{-\infty}^{(t/T)-n} g(xT)\exp\left[\left(\frac{\pi}{Q}f_oT - j2\pi f_oT\right)x\right]dx. \end{aligned} \tag{51}$$

In (49), $|A(t)|$ represents the amplitude modulation on the carrier, while $\Phi(t)$ represents the phase modulation, the quantity of primary interest here.

A-2. *Equation for normalized timing error*

There is no loss in generality and it is convenient if the timing error is evaluated in the neighborhood of the pulse that occurs for $n = 0$.

In this neighborhood, negative-going zero crossings occur where

$$2\pi f_r t + \Phi(t) = \frac{\pi}{2}$$

or

$$\frac{t}{T} = \frac{1}{4} - \frac{\Phi(t)}{2\pi}. \tag{52}$$

Similarly, positive-going zero crossings occur for

$$\frac{t}{T} = -\frac{1}{4} - \frac{\Phi(t)}{2\pi}. \tag{53}$$

In the absence of tuning error, and with impulse excitation, $\Phi = 0$ and the negative and positive-going zero crossings occur close to $\pm T/4$ respectively.* Using these zero crossings as a reference, it is easily seen that the equations for normalized timing error become

$$\frac{e_1}{T} = -\frac{\Phi\left(\frac{1}{4} + \frac{e_1}{T}\right)}{2\pi} \tag{54}$$

for negative-going zero crossings and

$$\frac{e_2}{T} = -\Phi\left(-\frac{1}{4} + \frac{e_2}{T}\right) \tag{55}$$

for positive-going zero crossings.

With the exception of the minor generalization to arbitrary pulse shape, the method employed thus far is identical with that used by Rowe.[2] At this point in the evaluation of the timing error, we depart from his approximate solutions of (54) and (55) and attempt other approaches. Before proceeding in this direction, an indication of the approximation used by Rowe will be given. For the high Q case, Φ will be small and will change only a small amount for small changes in $2\pi f_r t$. Based on this assumption,

$$\begin{aligned} \frac{e_1}{T} &\doteq -\frac{\Phi(\frac{1}{4})}{2\pi}, \\ \frac{e_2}{T} &\doteq -\frac{\Phi(-\frac{1}{4})}{2\pi}. \end{aligned} \tag{56}$$

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* Neglecting $\tan^{-1} \dfrac{1}{2Q}$ in (50)

It should be pointed out that these initial approximations are good for Rowe's purposes (steady-state error for $1/M$ patterns). However, for our purposes they need to be improved.

A-3. *Approximate solution of equation for normalized timing error*

One method for improving the accuracy of the initial approximation is to expand Φ in a power series about $T/4$ for negative-going zero crossings and retain two terms in the expansion to get

$$\frac{e_1}{T} = -\frac{\Phi(\frac{1}{4})}{2\pi + \Phi'(\frac{1}{4})}. \tag{57}$$

The form of Φ makes this approach messy and makes the determination of the probability distribution more difficult.

Another approach that is more tractable involves the separate Taylor expansion of I_1 and I_2 (51) in Φ about the reference time. If we retain only the first two terms in the Taylor expansion, replace the arctangent by its argument, and neglect k with respect to unity, we obtain for negative-going zero crossings

$$\frac{e_1}{T} = -\frac{1}{4\pi Q} - \frac{k}{4} - \frac{1}{2\pi}\,\frac{\begin{array}{l}\sum_{-\infty}^{\infty} a_n e^{(\pi/Q)n}\,[-\sin 2\pi kn\,(I_1(\frac{1}{4}-n) + e_1 I_1'(\frac{1}{4}-n)) \\ \qquad\qquad + \cos 2\pi kn(I_2(\frac{1}{4}-n) + e_1 I_2'(\frac{1}{4}-n))]\end{array}}{\begin{array}{l}\sum_{-\infty}^{\infty} a_n e^{(\pi/Q)n}[\cos 2\pi kn\,(I_1(\frac{1}{4}-n) + e_1 I_1'(\frac{1}{4}-n)) \\ \qquad\qquad + \sin 2\pi kn\,(I_2(\frac{1}{4}-n) + e_1 I_2'(\frac{1}{4}-n))]\end{array}}. \tag{58}$$

If terms in $(e_1/T)^2$ are neglected, multiplication of both sides of (58) by the long denominator on the right results in a linear equation for e_1/T. This equation is applicable to arbitrary pulse shape, time-limited or not, and has been applied by one of the authors to periodic patterns of both Gaussian and raised cosine pulses in unpublished work. The results were compared with digital computer simulation and were in excellent agreement, thereby giving us confidence in using this approach for random pulse patterns. In this paper, we will concentrate on raised cosine pulses. This enables us to make use of some of the results given by H. E. Rowe in Section 2.5 of his paper.[2] For these time-limited pulses, the limits of integration on the I's of (51) are modified in an obvious way, and the upper limit on the sum over n is limited to the pulse im-

mediately succeeding the time slot of interest at $n = 0$ for negative-going zero crossings. The evaluation of the various I's required is discussed in Appendix B.

Subject to the above conditions, the normalized timing error, as derived in Appendix B, can be written in the following form:

$$\frac{e_1}{T} = \frac{Ay + Bx + C}{Dy + Ex + F}, \tag{59}$$

where

$$\begin{aligned} y &\equiv \sum_{n=0}^{\infty} a_n e^{-(\pi/Q)n} \sin 2\pi kn, \\ x &\equiv \sum_{n=0}^{\infty} a_n e^{-(\pi/Q)n} \cos 2\pi kn, \end{aligned} \tag{60}$$

and $a_o \equiv 1$ (a pulse definitely occurs for $n = 0$). A through F are defined in Appendix B and are functions of the pulse width and Q and mistuning of the tuned circuit. In addition, C and F are functions of the presence or absence of a pulse in the succeeding time slot for negative-going zero crossings if sufficient pulse overlap exists. For positive-going zero crossings the form of the equation for the normalized timing error is the same and the new C and F are dependent upon the presence or absence of a pulse in the preceding time slot. This assumes that the pulse width is less than $2.5T$.

A-4. *Modification of probability distributions for pulse overlaps*

With the dependence on the occurrence of a succeeding pulse, as is the case for negative-going zero crossings with sufficient pulse overlap, we must modify the determination of the probability distribution as given in the main body of the paper. If we denote e_{11}/T and $C = C_1$, $F = F_1$ for $a_1 = 1$ (a succeeding pulse definitely occurs), and denote e_{12}/T and $C = C_2$, $F = F_2$ for $a_1 = 0$, then the average probability distribution for the timing deviation will be given by

$$\text{Prob}\left(\frac{e_1}{T} \leqq \lambda\right) = p \text{ Prob}\left(\frac{e_{11}}{T} \leqq \lambda\right) + (1 - p) \text{ Prob}\left(\frac{e_{12}}{T} \leqq \lambda\right). \tag{61}$$

When the pulse width is less than $1.5T$, $C_1 = C_2$, $F_1 = F_2$, and therefore $e_{11} = e_{12}$ and the above modification is not required. A similar procedure is applicable for positive-going zero crossings.

APPENDIX B. RAISED COSINE PULSES

B-1. *Determination of I's*

For a raised cosine pulse centered at the origin and of width T/s, I of equation (51) becomes

$$I(x) = 0 \qquad x < -\frac{1}{2s}$$

$$I(x) = \int_{(1/2s)}^{x} (1 + \cos 2\pi s x_1) e^{[(\pi/Q) - j2\pi]Kx_1} dx_1 \qquad |x| \leqq \frac{1}{2s} \tag{62}$$

$$I(x) = I\left(\frac{1}{2s}\right) \qquad x > \frac{1}{2s}$$

where

$$K \equiv (1 + k)$$

The integral in (62) is readily evaluated to give

$$\begin{aligned} -jI = \frac{1}{2\pi K} & \left[\frac{e^{[(\pi/Q)-j2\pi]Kx} - e^{-[(\pi/Q)-j2\pi]K/2s}}{\left(1 + \frac{j}{2Q}\right)} \right] \\ & + \frac{1}{4\pi} \left[\frac{e^{[(\pi/Q)-j2\pi]Kx} e^{+j2\pi sx} - e^{-[(\pi/Q)-j2\pi]K/2s}}{(K - s) + j\frac{K}{2Q}} \right] \\ & + \frac{1}{4\pi} \left[\frac{e^{[(\pi/Q)-j2\pi]Kx} e^{-j2\pi sx} - e^{-[(\pi/Q)-j2\pi]K/2s}}{(K + s) + j\frac{K}{2Q}} \right] \end{aligned} \tag{63}$$

The derivatives required in the evaluation of (58) may be obtained from

$$\frac{dI}{dx} = e^{(\pi/Q)Kx} [e^{-j2\pi Kx} + \tfrac{1}{2} e^{-j2\pi(K-s)x} + \tfrac{1}{2} e^{-j2\pi(K+s)x}]. \tag{64}$$

In the evaluation of I and dI/dx, mistuning makes very little difference for the allowable values in practical systems. Therefore, with $K = 1$

$$I']_{x=1/4} = -j\, e^{\pi/4Q} \left[1 + \cos \frac{\pi s}{2} \right] \tag{65}$$

$$I']_{x=-1/4} = j\, e^{\pi/4Q} \left[1 + \cos \frac{\pi s}{2} \right] \tag{66}$$

$$I']_{x=3/4} = j\, e^{\pi/4Q}\left[1 + \cos\frac{3\pi s}{2}\right] \tag{67}$$

$$I']_{x=-3/4} = -j\, e^{\pi/4Q}\left[1 + \cos\frac{3\pi s}{2}\right]. \tag{68}$$

Equations (65) and (68) above are required for negative-going zero crossings, while (66) and (67) are needed for positive-going zero crossings.

B-2. *Equation for Normalized Timing Error with Raised Cosine Pulses*

From (58) we can write the equation for normalized timing error as

$$\frac{e_1}{T} \doteq -\frac{1}{4\pi Q} - \frac{k}{4} - \frac{1}{2\pi}\frac{N}{P}, \tag{69}$$

where N and P are defined by comparison with (58). Cross multiplication by P, neglecting terms in e_1^2 and collecting terms, yields

$$\frac{e_1}{T} = \frac{Ay + Bx + C}{Dy + Ex + F}, \tag{70}$$

where x and y are defined by (60), and A through F are as follows:

$$A = -\frac{1}{2\pi} I_1\left(\frac{1}{2s}\right) + \left[\frac{1}{4\pi Q} + \frac{k}{4}\right] I_2\left(\frac{1}{2s}\right)$$

$$B = -\frac{1}{2\pi} I_2\left(\frac{1}{2s}\right) - \left[\frac{1}{4\pi Q} + \frac{k}{4}\right] I_1\left(\frac{1}{2s}\right)$$

$$\begin{aligned} C = &-\frac{1}{2\pi}\left[I_2\left(\frac{1}{4}\right) - I_2\left(\frac{1}{2s}\right)\right] - \left[\frac{1}{4\pi Q} + \frac{k}{4}\right]\left[I_1\left(\frac{1}{4}\right) - I_1\left(\frac{1}{2s}\right)\right] \\ &+ \underline{a}_1\, e^{(\pi/Q)}\left\{\frac{1}{2\pi}\left[\sin 2\pi k I_1\left(-\frac{3}{4}\right) - \cos 2\pi k I_2\left(-\frac{3}{4}\right)\right]\right. \\ &\left. - \left[\frac{1}{4\pi Q} + \frac{k}{4}\right]\left[\sin 2\pi k I_2\left(-\frac{3}{4}\right) + \cos 2\pi k I_1\left(-\frac{3}{4}\right)\right]\right\} \end{aligned}$$

$$D = -I_2\left(\frac{1}{2s}\right)$$

$$E = I_1\left(\frac{1}{2s}\right)$$

$$F = I_1\left(\frac{1}{4}\right) - I_1\left(\frac{1}{2s}\right) + \left[\frac{1}{4\pi Q} + \frac{k}{4}\right] I_1'\left(\frac{1}{4}\right) + \frac{1}{2\pi} I_2'\left(\frac{1}{4}\right)$$

$$+ \underline{a}_1 e^{(\pi/Q)} \left\{ \cos 2\pi k \left[\frac{1}{2\pi} I_2' \left(-\frac{3}{4}\right) + I_1\left(-\frac{3}{4}\right) \right.\right.$$
$$\left. + \left(\frac{1}{4\pi Q} + \frac{k}{4}\right) I_1'\left(-\frac{3}{4}\right)\right] + \sin 2\pi k \left[-\frac{1}{2\pi} I_1'\left(-\frac{3}{4}\right)\right.$$
$$\left.\left. + I_2\left(-\frac{3}{4}\right) + \left(\frac{1}{4\pi Q} + \frac{k}{4}\right) I_2'\left(-\frac{3}{4}\right)\right]\right\}.$$

For positive-going zero crossings, only the constants C and F are changed.

B-3. *Numerical Evaluation of Constants*

In order to make use of some of Rowe's results, we will choose the same two cases for pulse width that he used.

Case 1. $s = 1$, *Pulses Resolved*

a. *Negative-Going Zero Crossings.* Since mistuning has a small effect on the evaluation of the I's, we neglect it in this regard. Neglecting terms in $1/Q^2$ and k/Q, after some arithmetic one arrives at

$$\frac{e_1}{T} = -\frac{\frac{1}{4\pi} y - \left(\frac{1}{16\pi Q} - \frac{k}{8}\right) x + \frac{0.0795}{2\pi} + \frac{0.0316}{Q} + 0.0085k}{\frac{3}{16\pi Q} y + \frac{1}{2}(x - 1) + 0.375 + \frac{0.06}{Q}}. \tag{71}$$

$Q > 50$ and $kQ < 0.2$ encompass values of practical interest. In this region the term in y in the denominator of (71) can be neglected and the numerator term 0.0085k is also negligible. It is also convenient to deal with phase error rather than timing error. Therefore, we rewrite (71) as

$$\theta_1 = -\frac{2\pi e_1}{T} = \frac{y - \frac{1}{Q}\left[\frac{1}{4} - \frac{\pi}{2} kQ\right] x + 0.159 + \frac{0.397}{Q}}{x - 0.25 + \frac{0.12}{Q}}. \tag{72}$$

The multiplication by -2π is used to avoid any questions later on as to which way certain inequalities are to be taken. This means that θ is the negative of the phase error as previously defined. A positive value of θ signifies that the zero crossing occurs prior to $\pm T/4$ for negative going and positive going zero crossings respectively. The general form of θ for all the cases to be considered herein then can be written as

$$\theta = \frac{y + a}{x + b} + c. \tag{73}$$

For the situation under consideration in this section,

$$a = 0.159 + \frac{0.334}{Q} + \frac{\pi}{8} k$$

$$b = -0.25 + \frac{0.12}{Q}$$

$$c = -\frac{1}{Q}\left[\frac{1}{4} - \frac{\pi}{2} kQ\right].$$

b. *Positive-Going Zero Crossings.* Proceeding in the same way as in Sections B-2 and B-3 above, the phase error for positive-going zero crossings is as in (73) with

$$a = 0.159 - \frac{0.2}{Q} + \frac{3\pi}{8} k$$

$$b = -0.75 + \frac{0.62}{Q}$$

$$c = -\frac{1}{Q}\left[\frac{1}{4} - \frac{\pi}{2} kQ\right].$$

In this case it should be noted that with zero mistuning ($y = 0$) and with a pulse for $n = 0$ and nowhere else, a positive-going zero crossing does not occur in the neighborhood of $-T/4$. Under this special condition, $x = 1$ and (73) with the constants of this section would predict an incorrect error in the positive-going zero crossing. Of course such a sparse pattern occurs with probability zero. Fortunately, for all other more reasonable periodic patterns, results obtained from (73) are in good agreement with computer simulation.

Case 2. $s = \frac{2}{3}$, Pulses Overlapping, Base Width = 1.5T

a. *Negative-Going Zero Crossings.* In this section we will dispense with all of the algebra and arithmetic and simply write down the final results. For the case at hand

$$\frac{e_1}{T} \doteq -\frac{\dfrac{0.255}{2\pi} y - \dfrac{1}{Q}(0.073 - 0.064kQ)x + 0.0264 + \dfrac{1}{Q} \cdot (0.034 - 0.02kQ)}{0.255x - 0.062 + 0.048/Q}. \tag{74}$$

When this is converted to the form of (73), we have

$$a = 0.65 + \frac{0.4}{Q} - 0.21k$$

$$b = -0.243 + \frac{0.188}{Q}$$

$$c = -\frac{1}{Q}[1.8 - 1.58kQ].$$

b. *Positive-Going Zero Crossings*

$$a = 0.65 - \frac{0.4}{Q} + 0.94k$$

$$b = -0.753 + \frac{1.66}{Q}$$

$$c = -\frac{1}{Q}[1.8 - 1.58kQ].$$

The remarks made in connection with positive-going zero crossings for Case 1 are equally applicable here.

APPENDIX C. SEMI-INVARIANTS FOR THE JOINT DENSITY FUNCTION OF x_1 AND y_1

C-1. *One out of M pulses definitely occur; the remaining pulses are independent and occur with probability $\frac{1}{2}$; raised cosine pulses.*

The characteristic function is defined as

$$\varphi(u,v) = E \exp i(ux_1 + vy_1), \tag{75}$$

where E is the expectation operator, and from Appendix B

$$\begin{aligned} x_1 &\equiv \sum_{m=0}^{\infty} e^{-\alpha Mm} \cos 2\pi kMm + b + \sum_{n \neq mM}^{\infty} a_n e^{-\alpha n} \cos 2\pi kn, \\ y_1 &\equiv \sum_{m=0}^{\infty} e^{-\alpha Mm} \sin 2\pi kMm + a + \sum_{n \neq mM}^{\infty} a_n e^{-\alpha n} \sin 2\pi kn, \end{aligned} \tag{76}$$

with $\alpha \equiv \pi/Q$. Substituting (76) in (75) and performing the expectation operation gives

$$\varphi(u,v) = \exp i \sum_{m=0}^{\infty} e^{-(\pi/Q)Mm} (u \cos 2\pi kMm + v \sin 2\pi kMm)$$

$$\cdot \exp i(ub + va) \times \prod_{n \neq mM}^{\infty} \exp\left\{\frac{i}{2} e^{-(\pi/Q)n}(u \cos 2\pi kn + v \sin 2\pi kn)\right\} \quad (77)$$

$$\times \prod_{n \neq mM}^{\infty} \cos\left\{\frac{e^{-(\pi/Q)n}}{2} (u \cos 2\pi kn + v \sin 2\pi kn)\right\}$$

which may be rearranged to

$$\varphi(u.v) = \exp\left[\frac{i}{2} \sum_{n=0}^{\infty} \left\{e^{-(\pi/Q)Mn} (u \cos 2\pi kMn + v \sin 2\pi kMn)\right.\right.$$

$$\left.\left. + e^{-(\pi/Q)n} (u \cos 2\pi kn + v \sin 2\pi kn)\right\}\right] \exp i(ub + va)$$

$$\times \frac{\prod_{n=0}^{\infty} \cos\left\{\frac{e^{-(\pi/Q)n}}{2} (u \cos 2\pi kn + \sin 2\pi kn)\right\}}{\prod_{n=0}^{\infty} \cos\left\{\frac{e^{-(\pi/Q)Mn}}{2} (u \cos 2\pi kMn + v \sin 2\pi kMn)\right\}}. \quad (78)$$

When we take the logarithm of (78), we obtain

$$\log \varphi(u,v) = \frac{i}{2} \sum_{n=0}^{\infty} [\beta^{Mn}(u \cos 2\pi kMn + v \sin 2\pi kMn)$$

$$+ \beta^{n}(u \cos 2\pi kn + v \sin 2\pi kn)]$$

$$+ i(ua + vb) + \sum_{n=0}^{\infty} \log \cos\left[\frac{\beta^n}{2} (u \cos 2\pi kn + v \sin 2\pi kn)\right] \quad (79)$$

$$- \sum_{n=0}^{\infty} \log \cos\left[\frac{\beta^{Mn}}{2} (u \cos 2\pi kMn + v \sin 2\pi kMn)\right],$$

where $\beta = e^{-(\pi/Q)}$.

The first sum in (79) may be carried out, and when combined with $i(ua + vb)$ yields the semi-invariants λ_{10} and λ_{01} which are of course the mean values for x_1 and y_1 respectively. Since the last two terms of (79) are similar in form, we will confine our manipulations to the next to the last term. We denote this term by

$$F(u,v) = \sum_{n=0}^{\infty} \log \cos\left[\frac{\beta^n}{2} (u \cos 2\pi kn + v \sin 2\pi kn)\right]. \quad (80)$$

Using the infinite product expansion for the cosine and the power series expansion for the log; i.e.,

$$\cos z = \prod_{m=0}^{\infty} \left[1 - \left(\frac{2z}{(2m+1)\pi} \right)^2 \right] \qquad (z^2 < \infty)$$

and

$$\log (1 - x) = - \sum_{j=1}^{\infty} \frac{x^j}{j} \qquad (x^2 < 1).$$

$F(u,v)$ becomes

$$F(u,v) = - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{j=1}^{\infty} \frac{(uC_n + vS_n)^{2j}}{i(2m+1)^{2i}\pi^{2i}}, \tag{81}$$

where $C_n \equiv e^{-\alpha n} \cos 2\pi kn$ and $S_n \equiv e^{-\alpha n} \sin 2\pi kn$. The sum over j may be obtained by virtue of

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^{2i}} = \frac{(2^{2j} - 1)(-1)^{j-1}(2\pi)^{2j} B_{2j}}{2^{2j+1}(2j)!}$$

where the B_{2j} are the Bernoulli numbers. With the above sum over m and the expansion of $(uC_n + vS_n)^{2j}$ in a binomial series, we arrive at

$$F(u,v) = \sum_{j=1}^{\infty} \frac{(-1)^j B_{2j}(2^{2j} - 1)}{2j(2j)!} \sum_{r=0}^{2j} \binom{2j}{r} \sum_{n=0}^{\infty} C_n^r u^r (S_n v)^{2j-r} \tag{82}$$

Proceeding in the same manner that took us from (80) to (82), it can be verified that the last term of (79) takes the same form as the right-hand side of (82) with n replaced by nM. These results and comparison with the definition of the semi-invariants for a two dimensional distribution[10] lead to the following for the semi-invariants for the process under consideration:

$$\lambda_{10} = \frac{1}{2} \left[\frac{1 - \beta \cos 2\pi k}{1 - 2\beta \cos 2\pi k + \beta^2} + \frac{1 - \beta^M \cos 2\pi kM}{1 - 2\beta^M \cos 2\pi kM + \beta^{2M}} \right] + b,$$

$$\lambda_{01} = \frac{1}{2} \left[\frac{\beta \sin 2\pi b}{1 - 2\beta \cos 2\pi k + \beta^2} + \frac{\beta^M \sin 2\pi kM}{1 - 2\beta^M \cos 2\pi kM + \beta^{2M}} \right] + a,$$

and

$$\lambda_{rs}]_{r+s>1} = \frac{B^{r+s}(2^{r+s} - 1)}{r + s} \sum_{n=0}^{\infty} [C_n^r S_n^s - C_{nM}^r S_{nM}^s]. \tag{83}$$

The sum over n can be shown to be a geometric series multiplied by two finite series if the sines and cosines in S and C respectively are represented in exponential form and use is made of the binomial expansion. After some algebra, an alternate form for (83) can be shown to be

$$\lambda_{rs}]_{r+s>1} = \frac{B_{r+s}(2^{r+s} - 1)r!\,s!}{(r + s)2^r(2i)^s}\, G(r,s,\beta,k,M), \tag{84}$$

where $G(r,s,\beta,k,M)$ is (shortened to G)

$$\begin{aligned} G = \sum_{p=0}^{r} \sum_{q=0}^{s} & \frac{(-1)^q}{q!(r-p)!q!(s-q)!} \\ & \cdot \left[\frac{1}{1 - \beta^{(r+s)} \exp{[i2\pi k(r + s - 2p - 2q)]}} \right. \\ & \left. - \frac{1}{1 - \beta^{M(r+s)} \exp{[i2\pi kM(r + s - 2p - 2q)]}} \right]. \end{aligned} \tag{85}$$

For u and v in the neighborhood of zero, the contributions to the series in (79) become smaller as n becomes larger. The importance of successive terms is judged by the exponential decay factor $e^{-(n\pi/Q)}$. If we consider all terms up to some $n_{\max}$ where $n_{\max} \gg Q/\pi$ and $kn_{\max} \ll 1$, then we arrive at the following inequality

$$\frac{kQ}{\pi} \ll 1. \tag{86}$$

Under the above condition $\cos 2\pi kn$ can be replaced by unity and $\sin 2\pi kn$ by $2\pi kn$ for all terms of importance in the series and (79) becomes approximately

$$\begin{aligned} \log \varphi(u,v) \sim i & \left\{ \frac{u}{2}\left[\frac{1}{1-\beta} + \frac{1}{1-\beta^M}\right] + v\pi k \left[\frac{\beta}{(1-\beta)^2} + \frac{M\beta^M}{(1-\beta^M)^2}\right] \right. \\ & \left. + (ub + va) \right\} + \sum_{n=0}^{\infty} \log \cos \left\{ \frac{\beta^n}{2}(u + 2\pi knv) \right\} \\ & - \sum_{n=0}^{\infty} \log \cos \left\{ \frac{\beta^{Mn}}{2}(u + 2\pi kMnv) \right\}. \end{aligned} \tag{87}$$

Paralleling the operations performed on (80) to obtain (82) it can be shown that the semi-invariants obtained from (87) under the condition (86) are

$$\begin{aligned} \lambda_{10} &= \frac{1}{2}\left(\frac{1}{1-\beta} + \frac{1}{1-\beta^M}\right) + b, \\ \lambda_{01} &= \pi k \left(\frac{\beta}{(1-\beta)^2} + \frac{M\beta^M}{(1-\beta^M)^2}\right) + a, \quad \text{and} \\ \lambda_{rs}]_{r+s>1} &= (-1)^s \frac{B_{r+s}(2^{r+s} - 1)}{(r+s)} (2\pi k)^s \frac{d^s}{dg^s}\left(\frac{1}{1-e^{-g}} - \frac{1}{1-e^{-Mg}}\right), \end{aligned} \tag{88}$$

with $g = (r + s)\pi/Q$.

C-2. *Same as I Above Except That Pulses are Impulses*

For this case the semi-invariants are as above with $a = 0 = b$.

C-3. *Impulse Excitation, All Pulses Random*

With this type of excitation, we have

$$\lambda_{10} = \frac{1}{2}\left[\frac{1 - \beta\cos 2\pi k}{1 - 2\beta\cos 2\pi k + \beta^2}\right],$$

$$\lambda_{01} = \frac{1}{2}\left[\frac{\beta\sin 2\pi k}{1 - 2\beta\cos 2\pi k + \beta^2}\right],$$

and

$$\begin{aligned}\lambda_{rs}\big]_{r+s>1} = \frac{B_{r+s}(2^{r+s}-1)_{r!s!}}{(r+s)2^r(2i)^s}\Bigg[\sum_{p=0}^{r}\sum_{q=0}^{s}\frac{(-1)^q}{p!(r-p)!q!(s-q)!} \\ \cdot\frac{1}{1-\beta^{r+s}\exp\left[i2\pi k(r+s-2p-2q)\right]}\Bigg].\end{aligned} \tag{89}$$

It is readily shown in this case that the approximate semi-invariants [subject to (86)] are

$$\begin{aligned}\lambda_{10} &= \frac{1}{2}\left(\frac{1}{1-\beta}\right), \\ \lambda_{01} &= \frac{\pi k\beta}{(1-\beta)^2} \\ \lambda_{rs}\big]_{r+s>1} &= (-1)^s\,\frac{B_{r+s}(2^{r+s}-1)}{r+s}\,(2\pi k)^s\,\frac{d^s}{dg^s}\left(\frac{1}{1-e^{-g}}\right),\end{aligned} \tag{90}$$

with $g = (r + s)\pi/Q$.

APPENDIX D

High Q Behavior of $p(\theta)$

To illustrate the behavior of the probability density function when the Q of the resonator becomes large, we consider $p(\theta)$ in the neighborhood of the mean, θ_o. We include terms of the double summation in (19) for which $r + s = 4$. Since the Bernoulli numbers $B_{r+s} = 0$ for $r + s$ odd and >1, the terms λ_{rs} for $r + s = 3$ are zero. For $\theta \sim \theta_o$, therefore, $p(\theta)$ becomes

$$p(\theta) \doteq \frac{1}{\sqrt{2\pi}} \frac{\lambda_{10}}{(\lambda_{20}\theta_o^2 - 2\lambda_{11}\theta_o + \lambda_{02})^{\frac{1}{2}}} \cdot \exp -\frac{\lambda_{10}^2}{2} \frac{(\theta - \theta_o)^2}{(\lambda_{20}\theta_o^2 - 2\lambda_{11}\theta_o + \lambda_{02})} \tag{91}$$

$$\cdot \left[1 + \frac{H_4\left(\dfrac{\lambda_{10}(\theta - \theta_o)}{\sqrt{2}(\lambda_{20}\theta_o^2 - 2\lambda_{11}\theta_o + \lambda_{02})^{\frac{1}{2}}}\right)}{\dfrac{\sqrt{2}(\lambda_{20}\theta_o^2 - 2\lambda_{11}\theta_o + \lambda_{02})^{\frac{1}{2}}}{\lambda_{10}}} \sum_{\substack{r \\ r+s>2}}^{r+s=4} \sum_{s} (-1)^r \frac{\lambda_{rs}}{r!s!} \frac{\theta_o^r}{\lambda_{10}^4} \right].$$

The semi-invariants of interest in the above equation are given below and were determined using the results of the previous section for the case "all impulses random," subject to $kQ \ll \pi$.

$$\lambda_{10} = \frac{1}{2(1-\beta)} \qquad \theta_o = \frac{\lambda_{01}}{\lambda_{10}} = \frac{2\pi k\beta}{1-\beta}$$

$$\lambda_{20} = \frac{1}{4} \frac{1}{(1-\beta^2)} \qquad \lambda_{11} = \frac{\pi k}{2} \frac{\beta^2}{(1-\beta^2)^2} \qquad \lambda_{02} = \frac{(\pi k)^2\beta^2(1+\beta^2)}{(1-\beta^2)^3}$$

$$\lambda_{40} = -\frac{1}{8} \frac{1}{(1-\beta^4)} \quad \lambda_{31} = -\frac{\pi k}{4} \frac{\beta^4}{(1-\beta^4)^2} \quad \lambda_{22} = -\frac{(\pi k)^2}{2} \frac{\beta^4(1+\beta^4)}{(1-\beta^3)^3}$$

$$\lambda_{13} = -(\pi k)^3 \frac{\beta^4}{(1-\beta^4)^4} (1 + 4\beta^4 + \beta^8)$$

$$\lambda_{04} = -2(\pi k)^4 \frac{\beta^4}{(1-\beta^4)^5} (1 + 11\beta^4 + 11\beta^8 + \beta^{12}).$$

Using the above expressions for the λ's, the following quantities in (91) may be reduced to

$$\frac{\lambda_{10}}{(\lambda_{20}\theta_o^2 - 2\lambda_{11}\theta_o + \lambda_{02})^{\frac{1}{2}}} = \frac{1}{\dfrac{\sqrt{2}(2\pi k)\beta}{(1-\beta)^{\frac{1}{2}}(1-\beta)^{\frac{3}{2}}}} = \frac{1}{\sigma},$$

$$\sum_{\substack{r \\ r+s>2}}^{r+s=4} \sum_{s} (-1)^r \frac{\lambda_{rs}}{r!s!} \frac{\theta_o^r}{\lambda_{10}^4} = -\frac{1}{4!} \frac{2(2\pi k)^4\beta^4}{1-\beta^4}$$

$$\cdot \left[1 - \frac{4\beta^3(1-\beta)}{(1-\beta^4)} + \frac{6\beta^2(1+\beta^4)(1-\beta)^2}{(1-\beta^4)^2} - \frac{4\beta(1-\beta)^3(1+4\beta^4+\beta^8)}{(1-\beta^4)^3} + \frac{(1-\beta)^4(1+11\beta^4+11\beta^8+\beta^{12})}{(1-\beta^4)^4} \right],$$

or

$$\sum_{\substack{r \\ r+s>2}}^{r+s=4} \sum_{s} (-1)^r \frac{\lambda_{rs}}{r!\,s!} \frac{\theta_o^{\,r}}{\lambda_{10}^{\,4}} \equiv \frac{\lambda_4}{4!}.$$

The probability density therefore takes the form

$$p(\theta) \doteq \frac{1}{\sqrt{2\pi}\sigma} \exp - \frac{(\theta-\theta_o)^2}{2\sigma^2} \left[1 + \frac{\lambda_4}{4!} \frac{H_4\left(\dfrac{\theta-\theta_o}{\sqrt{2}\sigma}\right)}{4\sigma^4} \right]. \tag{92}$$

This result is in the form of the standard Edgeworth approximation with θ_o, σ, and λ_4 the mean, the standard deviation and the 4th semi-invariant of the θ distribution, respectively. In the limit as Q becomes large ($\beta \rightarrow 1$) we approximate $1 - \beta$ by π/Q and

$$\sigma \rightarrow k\sqrt{\pi Q} \quad \theta_o \rightarrow 2kQ$$

The coefficient of the 4th Hermite polynomial approaches $-(5\pi/128Q)$. Equation (92) then indicates the approach to the normal law with the first correction term going as $1/Q$. The results for θ_o and σ correspond to those derived earlier by Bennett, Rice and others.

APPENDIX E

Determination of $\theta_{\max}$

For $kQ \ll \pi$, a good approximation for θ is (from Appendix B)

$$\theta = \frac{a + 2\pi k \sum_{n=0}^{\infty} a_n n \beta^n}{b + \sum_{n=0}^{\infty} a_n \beta^n}. \tag{93}$$

When $a_o = 1$, we have

$$\frac{\theta}{2\pi k} = \frac{\dfrac{a}{2\pi k} + \sum_{n=1}^{\infty} a_n n \beta}{1 + b + \sum_{n=1}^{\infty} a_n \beta^n}. \tag{94}$$

It is of interest to determine the pulse pattern that yields the maximum value of $\theta/2\pi k$. This is equivalent to the determination of a one-zero sequence of a_n's such that (94) is a maximum.

Assume that an initial pattern has been chosen such that $\theta/2\pi k = A_o/B_o$. If a single a_n is changed from zero to one (pulse added), then $\theta/2\pi k$ is changed to $(A_o + n\beta^n)/(B_o + \beta^n)$. Clearly, we should effect this conversion if

$$\frac{A_o + n\beta^n}{B_o + B^n} \geqq \frac{A_o}{B_o}$$

or

$$n \geqq \frac{A_o}{B_o}. \tag{95}$$

On the other hand if a one is changed to a zero (pulse removed), then $\theta/2\pi k$ will be increased if

$$\frac{A_o - n\beta^n}{B_o - \beta^n} > \frac{A_o}{B_o}$$

or

$$n < \frac{A_o}{B_o}. \tag{96}$$

The process is continued in this manner until all $a_n = 1$ for $n \geqq n_c$ and all $a_n = 0$ for $n < n_c$ (except a_o, which is constrained to be unity). n_c may be determined from the above process, since

$$n_c = \frac{\theta_{\max}}{2\pi k} = \frac{\dfrac{a}{2\pi k} + \sum_{n=n_c}^{\infty} n\beta^n}{1 + b + \sum_{n=n_c}^{\infty} \beta^n}, \tag{97}$$

which can be rearranged to

$$\frac{\beta^{n_c+1}}{(1 - \beta)^2} = -\frac{a}{2\pi k} + n_c(1 + b). \tag{98}$$

When a periodic pulse pattern of 1 out of every M pulses is forced, $\theta_{\max}$ is found in the same manner as above and the relationship between the various parameters to achieve this maximum is given by (15) of the main body of the paper.

REFERENCES

1. Bennett, W. R., B.S.T.J., **37,** Nov., 1958, p. 1501.
2. Rowe, H. E., B.S.T.J., **37,** Nov., 1958, p. 1543.

3. DeLange, O. E., B.S.T.J., **37**, Nov., 1958, p. 1455.
4. DeLange, O. E., and Pustelnyk, M., B.S.T.J., **37**, Nov., 1958, p. 1487.
5. Sunde, E. D., B.S.T.J., **36**, July, 1957, p. 891.
6. DeLange, B.S.T.J., **35**, Jan., 1956, p. 67.
7. Aaron, M. R., B.S.T.J., **41**, Jan., 1962, p. 99.
8. Mayo, J. S., B.S.T.J., **41**, Jan., 1962, p. 25.
9. Cramér, H., *Mathematical Methods of Statistics*, Princeton University Press, Princeton, N. J., 1946, p. 317 and p. 224.
10. Laning and Battin, *Random Processes in Automatic Control*, McGraw-Hill Book Co., New York, 1956, p. 61.